GEOMETRICAL COEFFICIENTS AND MEASURES OF NONCOMPACTNESS

By

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To my parents

PREFACE

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy in the University of Glasgow. It presents the results of research conducted by the author at the University of Glasgow during the academic years 1989–92.

With the exception of the instances indicated within the text and attributed there to the authors concerned, and of some work done in collaboration with Professor J. R. L. Webb, the results in this thesis are the original work of the author alone.

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CONTENTS

SUMMARY

This thesis studies various measures of noncompactness and some geometrical coefficients in metric or Banach spaces. These geometrical numbers are useful in the study of measures of noncompactness, some of which are interesting quantities in fixed point theory.

In Chapter 1, we give some definitions and known results for convenience and later use. Some of these results have been obtained very recently.

In Chapter 2, we study a very useful geometrical coefficient in fixed point theory —— the Lifschitz characteristic, which is also useful in the study of measures of noncompactness. We will compare this number with other interesting geometrical numbers, and estimate its values in certain spaces. Also some fixed point theorems which employ this quantity are given. This chapter contains the work of [WebZ-1] and part of the work of [WebZ-2].

In Chapter 3, the notion of normal structure and several normal structure coefficients in Banach space are studied. The notion of normal structure has proved to be a very useful one and various types of normal structure coefficients, such as $N(X)$, $WCS(X)$ and $D(X)$, have been well studied in recent years. We will give several equivalent definitions of normal structure. Via these new characterizations of normal structure, some other normal structure coefficients are defined. We connect these geometrical numbers with $N(X)$, $WCS(X)$ and $D(X)$, and use them to relate various measures of noncompactness. This chapter contains part of the work of [Z-2].

In Chapter 4, we give various connections between the set measure of noncompactness $\alpha(\Omega)$, the ball measure of noncompactness $\beta(\Omega)$ and the separation measure of noncompactness $\delta(\Omega)$. In general, $\alpha(\Omega)\leq\beta(\Omega)\leq 2\alpha(\Omega)$ and

$\delta(\Omega) \leq \alpha(\Omega) \leq \beta(\Omega) \leq 2\delta(\Omega)$ are the best possible inequalities. These measures are closely related to geometrical properties of the underlying space. We can improve on $\beta(\Omega) \leq 2\alpha(\Omega)$, $\beta(\Omega) \leq 2\delta(\Omega)$ and $\delta(\Omega) \leq \beta(\Omega)$ by employing some geometrical coefficients which we also study in this Chapter. The improved inequalities are best possible in certain spaces such as Hilbert, l^p and L^p spaces. As consequences, relations between various measures of noncompactness for operators, $\alpha(f)$, $\beta(f)$ and $\delta(f)$, are given. Part of this chapter has been published in [WebZ-2] and part is to be published in [Z-1].

In Chapter 5, we study $\alpha(T)$ and $\beta(T)$ when T is a linear operator. We relate $\beta(T)$ and $\beta(T^*)$ by using a geometrical number. This includes that $\beta(T)=\beta(T^*)$ in certain spaces such as Hilbert and l^p $(1<p<\infty)$ spaces. Also we study the ascent and descent of $I-T$ and the eigenvalues of T by using $\alpha(T)$ and $\beta(T)$. Part of this chapter presents part of the work of [Z-2].

The main results are Theorems 2.1.2, 2.2.4, 2.3.3, 3.1.4, 3.1.8, 3.2.5, 3.3.4, 4.1.3, 4.2.1, 4.2.7, 4.2.10, 4.3.2, 4.4.7, 4.4.8, 5.1.2, 5.3.1, 5.3.3 .

INTRODUCTION

The class of k-set contractive operators has proved to be a useful one in the study of nonlinear operators; see, for example, [De], [EdE], [Li]. This class is defined via the notion of the set measure of noncompactness of a set. Let Ω be a bounded set in a metric space M, the set measure of noncompactness of Ω is

$$\alpha(\Omega)=\inf\left\{d>0 : \Omega \text{ can be covered by a finite number of sets of diameter} \leq d\right\}.$$

A continuous operator f is said to be a k-set contraction if $\alpha(f(\Omega)) \leq k\alpha(\Omega)$ for all bounded sets Ω. Contractions, compact mappings and sums of these furnish examples of k-set contractions. For a k-set contraction f, let

$$\alpha(f)=\inf\{k \geq 0 : f \text{ is a } k\text{-set contraction}\}.$$

If, instead of using sets to cover Ω, balls of diameter d are used, one defines the ball measure of noncompactness denoted by $\beta(\Omega)$. Also the k-ball contractions and $\beta(f)$ for a k-ball contraction f may be defined. The two measures share similar properties, and $\alpha(\Omega) \leq \beta(\Omega) \leq 2\alpha(\Omega)$ are the best possible relations between them in general, as examples show.

These measures are closely related to geometrical properties of the underlying space and it is possible to improve on the inequality $\beta(\Omega) \leq 2\alpha(\Omega)$ in certain spaces. Indeed, Danes [Dan-1], [Dan-2] showed that a smaller constant suffices in Hilbert space and in uniformly convex Banach spaces. We have obtained sharp inequalities by employing geometrical coefficients which have previously been used in the study of fixed points of various mappings, see [WebZ-2], [Z-1]. We proved $\beta(\Omega) \leq 2\alpha(\Omega)/\kappa(M)$ in a metric space M, where $\kappa(M)$ is the Lifschitz characteristic of M; and also that $\beta(\Omega) \leq 2\alpha(\Omega)/WCS(X)$ in a reflexive Banach space X, where $WCS(X)$ is the weakly convergent sequence coefficient of X. Both $\kappa(M)$ and $WCS(X)$ are quantities of much interest in fixed point theory [Lif], [KrZ], [By-2], their values are in $[1, 2]$, and in certain spaces $\kappa(M) > 1$ or/and $WCS(X) > 1$. We will give the precise definitions of them later.

In this thesis, we study the Lifschitz characteristic $\kappa(M)$ and the weakly convergent sequence coefficient $WCS(X)$ and some numbers related to them. As well as studying the set and ball measures of noncompactness, we also study the so-called separation measure of noncompactness $\delta(\Omega)$ of a bounded set Ω in a metric space (M, ρ), which is defined by

$$\delta(\Omega) = \sup\left\{d \geq 0 : \text{there is a sequence } \{x_n\} \subseteq \Omega \text{ such that } \rho(x_n, x_m) \geq d \text{ if } m \neq n\right\}.$$

We prove $\delta(\Omega)$ is equal to $\sup\{\alpha(\Omega') : \Omega' \subseteq \Omega$ and Ω' α-minimal$\}$ if Ω is an infinite set. The important notion of α-minimal sets is due to Benavides [Ben-1]: an infinite bounded set A is called α-minimal if for any infinite subset B of A, $\alpha(A) = \alpha(B)$. Similarly one defines β-minimal sets. We will use the α and β-minimal sets and their properties very often in the thesis. Using the separation measure, k-δ-contractions and $\delta(f)$ for a k-δ-contraction f can be defined. We will give various relations between $\alpha(\Omega)$, $\beta(\Omega)$ and $\delta(\Omega)$ by employing certain geometrical coefficients, which we will also study. As consequences, relations between $\alpha(f)$, $\beta(f)$ and $\delta(f)$ are given. If T is a linear operator, we obtain more information about $\alpha(T)$ and $\beta(T)$.

In Chapter 1, we give some definitions and known results for convenience and later use. Some of these results have been obtained very recently.

In Chapter 2, we study the Lifschitz characteristic $\kappa(M)$ defined by Lifschitz [Lif] in a metric space (M, ρ) :

$$\kappa(M)=\sup\{b>0: \text{ there exists } a>1 \text{ such that for all } x, y\in M \text{ and all } r>0,\ \rho(x,y)>r \text{ implies there exists } z\in M \text{ such that } \overline{B}(x, br)\cap\overline{B}(y, ar)\subseteq\overline{B}(z, r)\}.$$

Obviously $1\leq\kappa(M)\leq 2$. Indeed in certain spaces, $\kappa(M)>1$. For a bounded, complete metric space M, and $k<\kappa(M)$, Lifschitz [Lif] proved that every uniformly k-Lipschitzian mapping $f: M\longrightarrow M$ (that is, for any $x, y\in M$ and each positive integer n, $\rho(f^n x, f^n y)\leq k\rho(x, y)$) has fixed points. This is one reason we study this number, but we also discovered its connection with measures of noncompactness. For a Banach space X, it is also worthwhile to study $\kappa_0(X)$ defined by

$$\kappa_0(X)=\inf\{\kappa(C): C \text{ a closed, bounded, convex subset of } X \text{ with } \operatorname{diam}(C)>0\},$$

since for $k<\kappa_0(X)$, the uniformly k-Lipschitzian mapping $f: C\longrightarrow C$ has fixed points.

We will give some relations between $\kappa(X)$, $\kappa_0(X)$ and other useful numbers in fixed point theory. For a Banach space X, we prove $\kappa_0(X)\leq N(X)$, where $N(X)$ is the normal structure coefficient of X:

$$N(X)=\inf\left\{\operatorname{diam}(C)/r(C, C): C \text{ a closed, bounded, convex subset of } X \text{ with } \operatorname{diam}(C)>0\right\},$$

where $r(C, C)=\inf_{x\in C}\left\{\sup_{y\in C}\|x-y\|\right\}$ is the Chebyshev radius of C. We will also give an example where $N(X)>1$ but $\kappa(X)=\kappa_0(X)=1$. We show that $\kappa(X)\geq\kappa_0(X)\geq 1/[1-\delta_X(1)]$, where $\delta_X(\varepsilon)$: $[0, 2]\longrightarrow[0, 1]$ is the modulus of convexity of X defined by:

$$\delta_X(\varepsilon)=\inf\left\{1-\|x+y\|/2: \|x\|\leq 1,\ \|y\|\leq 1,\ \|x-y\|\geq\varepsilon\right\}.$$

Both $N(X)$ and $\delta_X(\varepsilon)$ are interesting quantities in fixed point theorey [CaM], [GoeK-2], [GoeKT]. From above we see that if X is uniformly convex (that is, $\delta_X(\varepsilon)>0$ for every $\varepsilon>0$), $\kappa(X)\geq\kappa_0(X)>1$. We will prove $\kappa(H)=\kappa_0(H)=\sqrt{2}$ if H is a

Hilbert space. An inequality is established in the space L^p $(1<p<2)$ and we use it and other known inequalities to give lower bounds for the values of $\kappa(L^p)$ and $\kappa_0(L^p)$ $(1<p<\infty)$. We also give some fixed point theorems by employing the Lifschitz characteristic. The work of [WebZ-1] and part of the work of [WebZ-2] are included in this chapter.

In Chapter 3, we study the notion of normal structure and some normal structure coefficients. A Banach space X is said to have normal structure if for every closed, bounded and convex subset C of X with $\operatorname{diam}(C)>0$, $r(C, C)<\operatorname{diam}(C)$. Uniformly convex and uniformly smooth Banach spaces furnish examples of such spaces. If a Banach space X is uniformly convex or more generally has normal structure, then for any nonempty, weakly compact and convex subset C of X, every nonexpansive mapping $f: C \longrightarrow C$ has a fixed point, see **[Bro]**, **[Goh]**, **[Ki]**. To obtain fixed point theorems in more general spaces, Bynum **[By-2]** defined the usual normal structure coefficient $N(X)$ in a Banach space X and the weakly convergent sequence coefficient $WCS(X)$ in a reflexive Banach space :

$$WCS(X)=\inf\left\{\operatorname{diam}_a\{x_n\}/\inf_{z\in\overline{\operatorname{co}}\{x_n\}}\phi(z) : \{x_n\}\subseteq X \text{ weakly convergent but not strongly convergent}\right\}.$$

where $\phi(z)=\limsup_{n\to\infty}\|x_n-z\|$, $\operatorname{diam}_a\{x_n\}=\limsup_{n\to\infty}\left\{\|x_m-x_k\| : m\geq n,\ k\geq n\right\}$ is the asymptotic diameter of $\{x_n\}$. In recent years, $N(X)$, $WCS(X)$ and another number $D(X)$ defined by Maluta **[Mal]** have been well studied **[Am]**, **[Ben-3]**, **[By-2]**, **[Mal]**, **[Pr]**. For a Banach space X, if any of the three numbers is more than 1, X has normal structure. These numbers have proved to be useful in order to obtain fixed points of nonexpansive mappings and uniformly Lipschitzian mappings **[GoeK-1]**, **[By-2]**, **[CaM]**. They are also useful to compare the set and the ball measures of noncompactness **[WebZ-2]**, **[Z-1]**.

We will give several equivalent definitions of normal structure. Via these new characterizations of normal structure, we define some other normal structure coefficients which are related to $N(X)$, $WCS(X)$ and $D(X)$. One of the normal structure coefficients we define is

$$N_\beta(X)=\inf\Big\{2\mathrm{diam}(C)/\beta_C(C) : C \text{ a closed, bounded, convex and noncompact subset of } X\Big\},$$

where $\beta_C(C):=\Big\{2d>0: \text{ there are } x_1, \ldots, x_n \in C \text{ so that } C\subseteq \bigcup_{i=1}^{n} B(x_i, d)\Big\}$ is a variant of the ball measure of noncompactness, and $\beta_C(C)<2\mathrm{diam}(C)$ gives another characterization of normal structure. $1\le N_\beta(X)\le 2$, and in certain spaces $N_\beta(X)>1$. We prove that $N_\beta(X)>1$ implies that X is reflexive and we use this coefficient to relate $\alpha(\Omega)$, $\delta(\Omega)$ and $\beta(\Omega)$, in fact

$$\beta(\Omega)\le 2\alpha(\Omega)/N_\beta(X) \text{ and } \beta(\Omega)\le 2\delta(\Omega)/N_\beta(X).$$

Also we show $N_\beta(X)=WCS(X)$ for any reflexive Banach space. Part of the work of [Z-2] is included in this chapter.

In Chapter 4, we give various connection between $\alpha(\Omega)$, $\beta(\Omega)$ and $\delta(\Omega)$. Generally, $\delta(\Omega)\le\alpha(\Omega)\le\beta(\Omega)\le 2\delta(\Omega)$ are the best inequalities between them. As shown above, we see that it is possible to improve on the inequalities $\beta(\Omega)\le 2\alpha(\Omega)$ and $\beta(\Omega)\le 2\delta(\Omega)$ in certain spaces. We will give more improved inequalities of this kind. We prove that $\beta(\Omega)\le 2\alpha(\Omega)/W(X)$ and $\beta(\Omega)\le 2\delta(\Omega)/W(X)$ in a separable, reflexive Banach space X, where $W(X)$ is a modification of $WCS(X)$ by replacing $\mathrm{diam}_a\{x_n\}$ with $\mathrm{diam}\{x_n\}$, and replacing $\inf\limits_{z\in\overline{\mathrm{co}}\{x_n\}}\phi(z)$ with $\inf\limits_{z\in X}\phi(z)$ in the definition of $WCS(X)$. Obviously $W(X)\ge WCS(X)$, so $W(X)\ge N_\beta(X)$ in a reflexive Banach space. Also $W(X)\ge\kappa(X)$ is true (this will be proved in Chapter 2). Then we obtain better inequalities, but these are only true in separable spaces.

We define another number $K_\beta(X)$ in Banach space X by

$$K_\beta(X)=\inf\Big\{2\mathrm{diam}(C)/\beta(C) : C \text{ a closed, bounded, convex and noncompact subset of } X\Big\},$$

and we show that $\beta(\Omega)\le 2\alpha(\Omega)/K_\beta(X)$, which is the best possible inequality of this kind. For a Banach space X, let

$$K^0_\beta(X)=\inf\Big\{K_\beta(F)\text{: where } F \text{ is an infinite-dimensional, separable, closed subspace of } X\Big\}.$$

Then $\beta(\Omega)\le 2\delta(\Omega)/K^0_\beta(X)$ is true. We also obtain other inequalities of this kind by employing other geometrical numbers. We study these numbers and relate them in cetain spaces. Also the values of $W(X)$, $K_\beta(X)$, $K^0_\beta(X)$ are given in certain spaces such as Hilbert, l^p, L^p spaces.

We will also improve on $\beta(\Omega)\ge\delta(\Omega)$ in a Banach space X. We show that $\beta(\Omega)\ge\delta(\Omega)/\delta(\overline{\mathrm{B}}_{\mathrm{X}})$, where $\overline{\mathrm{B}}_{\mathrm{X}}$ is the closed unit ball of X. We will give some properties of $\delta(\overline{\mathrm{B}}_{\mathrm{X}})$, and show $\delta(\overline{\mathrm{B}}_{\mathrm{X}})<2$ in certain spaces. Since the values of $\delta(\overline{\mathrm{B}}_{\mathrm{X}})$ in l^p and L^p $(1<p<\infty)$ are known and less than 2, the inequality we obtain is sharp.

The results we obtain include that $\beta(\Omega)=\sqrt{2}\delta(\Omega)$ in Hilbert space; $\beta(\Omega)=2^{1-1/p}\delta(\Omega)$ in l^p $(1\le p<\infty)$ spaces; and

$$\min\{2^{1-1/p}, 2^{1/p}\}\delta(\Omega)\le\beta(\Omega)\le\max\{2^{1-1/p}, 2^{1/p}\}\delta(\Omega)$$

in L^p $(1<p<\infty)$ spaces. This recovers results in [Ben-1], [Ben-2] and [BenA].

As consequence, results about various contractive type mappings are proved. For $f: D(f)\subseteq X\longrightarrow Y$, where Y is also a Banach space, we obtain that

$$\beta(f)\le\big(\delta(\overline{\mathrm{B}}_{\mathrm{X}})/K^0_\beta(Y)\big)\delta(f)\le\big(\delta(\overline{\mathrm{B}}_{\mathrm{X}})/K^0_\beta(Y)\big)\alpha(f),$$

and if Y is separable, $\beta(f)\le\big(\delta(\overline{\mathrm{B}}_{\mathrm{X}})/K_\beta(Y)\big)\delta(f)\le\big(\delta(\overline{\mathrm{B}}_{\mathrm{X}})/K_\beta(Y)\big)\alpha(f)$; also we have $\delta(f)\le\alpha(f)\le[2/K_\beta(X)]\beta(f)$. The corollaries of above inequalities in Hilbert, l^p, L^p spaces contain results in [Ben-1], [Ben-2], [BenA] as special cases.

Part of the work of [WebZ-2] and [Z-1] is included in this chapter.

In Chapter 5, We study $\alpha(T)$ and $\beta(T)$ when T is a linear operator. If T is a bounded linear operator from a Banach space X to another Banach space Y, we know that both $\alpha(T)$ and $\beta(T)$ are seminorms on T. These seminorms have been well studied in recent years. Various relations between $\alpha(T)$, $\alpha(T^*)$, $\beta(T)$ and $\beta(T^*)$ have been established. Nussbaum [Nu-1] proved the important fact that $\alpha(T) \leq \beta(T^*)$ and $\alpha(T^*) \leq \beta(T)$. Webb [Web-1] and [Web-2] showed that $\alpha(T) = \beta(T)$ and $\beta(T^*) \leq \beta(T)$ in Hilbert spaces. Goldenstein and Markus [GolM] gave the relation between $\beta(T)$ and $\beta(T^*)$ as $\beta(T)/2 \leq \beta(T^*) \leq 2\beta(T)$. We continue this work and relate $\beta(T)$ and $\beta(T^*)$ by using a geometrical number $R_\beta(X)$, which is defined by :

$$R_\beta(X) = \sup\left\{\beta_C(C)/2 : C \subseteq \overline{B}_X \text{ is closed and convex}\right\}.$$

We will show that $\beta(T)/R_\beta(X) \leq \beta(T^*) \leq R_\beta(Y^*)\beta(T)$. Obviously $1 \leq R_\beta(X) \leq 2$. If X is uniformly convex or uniformly smooth, $R_\beta(X) < 2$. We are able to prove that $R_\beta(X) = 1$ in certain spaces such as Hilbert or l^p $(1 < p < \infty)$ spaces. Hence our results extend the inequalities $\beta(T)/2 \leq \beta(T^*) \leq 2\beta(T)$. We also obtain estimates for $R_\beta(X)$ in terms of $N_\beta(X)$ and other geometrical numbers in X.

For $T: X \longrightarrow X$, it is well known that $\alpha(T)$ and $\beta(T)$ are related to spectral properties of T especially the essential spectrum [Nu-1]. We give an elementary proof that $I - T$ has finite ascent and finite descent if $\alpha(T) < 1$ or $\beta(T) < 1$. For any $\lambda \in \mathbb{R}$, if $|\lambda| > \alpha(T)$, we prove that λ in the resolvent set of T or λ is an eigenvalue of T of finite algebraic multiplicity. Also for $r > \alpha(T)$, there are at most finitely many points λ in the spectrum of T with $|\lambda| \geq r$ and all such λ's are eigenvalues of T. This improves an argument of [Mar] where $r > 2\alpha(T)$ was needed.

Part of the work of [Z-2] is included in this chapter.

CHAPTER ONE

PRELIMINARIES

In this chapter, we provide some definitions and known results for convenience and later use. Some of these results are fundamental, others have been obtained very recently. To keep this chapter within reasonable length, we only state most of the results without proof, but the references are given.

1.1. Notation in Banach spaces

Let X be a Banach space, X^* its *dual space*, the value of $x^* \in X^*$ at $x \in X$ is denoted by (x^*, x) or $x^*(x)$. The norm on X will be denoted by $\|\cdot\|_X$ or by $\|\cdot\|$ if no ambiguity is possible. The *duality mapping* $F: X \to 2^{X^*}$ is defined by

$$F(x) = \left\{x^* \in X^* : \ (x^*, x) = \|x\|^2 = \|x^*\|^2\right\}.$$

where 2^{X^*} denotes the collection of all subsets of X^*. Using the Hahn–Banach theorem, it is clear that $F(x) \neq \emptyset$ for any $x \in X$. We use J_X to denote the *natural embedding* of X in its second dual space X^{**}. We use $\mathbb{R}$ to denote the set of real numbers, $\mathbb{R}^+$ the positive real numbers, $\mathbb{N}$ the positive integers.

Let M be a metric space with distance ρ. For any $x \in M$ and $r \in \mathbb{R}^+$, $B(x, r) := \{y \in M : \ \rho(x, y) < r\}$ denotes the open ball with centre x and radius r. For convenience, we also use B_X or B_1 to denote the unit ball $B(0, 1)$ if $M = X$ is a normed linear space. Let Ω be a subset of M, we use $\overline{\Omega}$ to denote the closure of Ω, $\operatorname{diam}(\Omega)$ the diameter of Ω, $\operatorname{dist}(x, \Omega) := \inf_{y \in \Omega} \rho(x, y)$ the distance from x to Ω. If M is a normed linear space, $\mathrm{co}\Omega$ denotes the convex hull of Ω, $\mathrm{span}(\Omega)$ the subspace spanned by Ω.

For a sequence $\{x_n\}$ in Banach space X, we write $w\text{-}\lim_{n\to\infty} x_n = x$ to denote that $\{x_n\}$ converges weakly to x.

For Banach spaces X and Y, we use $\mathscr{L}(X, Y)$ to denote the Banach space of all bounded linear operators from X to Y, with the usual norm. We write $\mathscr{L}(X)$ instead of $\mathscr{L}(X, X)$. Given any $T \in \mathscr{L}(X, Y)$, the *adjoint of* T is the operator $T^* \in \mathscr{L}(Y^*, X^*)$ defined by $(T^*y^*, x) = (y^*, Tx)$ for all $y^* \in Y^*$ and $x \in X$. Hence $T^{**}J_X = J_Y T$.

1.2. Some geometrical coefficients

The classes of Banach spaces which have some good geometrical properties are in some sense "nice" since important results can be obtained in these kind of spaces. In this section, we introduce some geometrical notions which have proved useful in fixed point theory and in the study of measures of noncompactness.

Definition 1.2.1. The *modulus of convexity* of a Banach space X is the function $\delta_X : [0, 2] \longrightarrow [0, 1]$ defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \|x+y\|/2 : \|x\| \le 1, \ \|y\| \le 1, \ \|x-y\| \ge \varepsilon\right\}.$$

X is said to be *uniformly convex* if $\delta_X(\varepsilon) > 0$ for every $\varepsilon > 0$, and X uniformly smooth if X^* is uniformly convex. X is said to be *strictly convex* if $\delta_X(2) = 1$.

$\delta_X(\varepsilon)$ can also be defined as: $\inf\left\{1 - \|x+y\|/2 : \ \|x\| = \|y\| = 1, \ \|x-y\| \ge \varepsilon\right\}$

(see [Day]). Clarkson [Cl] showed that Hilbert and l^p, L^p $(1<p<\infty)$ spaces are uniformly convex. Also he gave

$$\delta_p(\varepsilon)\geq 1-[1-(\varepsilon/2)^p]^{1/p} \text{ for } p\geq 2, \quad \text{and } \delta_p(\varepsilon)\geq 1-[1-(\varepsilon/2)^q]^{1/q} \quad \text{for } 1<p<2,$$

where $\delta_p(\varepsilon)$ denotes $\delta_{l^p}(\varepsilon)$ or $\delta_{L^p}(\varepsilon)$, and $q=1-1/p$. Hanner [Ha] later proved that equality holds for $p\geq 2$ and established the following precise implicit formula for $1<p<2$:

$$[1-\delta_p(\varepsilon)+\varepsilon/2]^p+|1-\delta_p(\varepsilon)-\varepsilon/2|^p=2.$$

Obviously, every uniformly convex Banach space is strictly convex. X is strictly convex if, and only if, $\|x\|\leq 1$, $\|y\|\leq 1$ and $\|x-y\|>0$ implies $\|x+y\|/2<1$ ([GoeK-1], Lemma 5.2). If X^* is strictly convex, the duality mapping F is single-valued [Bar], p.13.

Next we state a Lemma on a uniformly convex Banach space, which is an easy consequence of uniform convexity. We will use it in Chapter 2.

<u>Lemma 1.2.2.</u> (Dotson, [Dot]) *Suppose X is a uniformly convex Banach space. Let $a, b\in\mathbb{R}$ satisfy $0<a<b<1$, and let $\{c_n\}$ is a sequence in $[a, b]$. Suppose $\{w_n\}$ and $\{y_n\}$ are sequences in X such that $\|w_n\|\leq 1$, $\|y_n\|\leq 1$ for all n. Define $\{z_n\}$ in X by $z_n=(1-c_n)w_n+c_ny_n$. If $\lim_{n\to\infty}\|z_n\|=1$, then $\lim_{n\to\infty}\|w_n-y_n\|=0$.*

The following Lemma is on the duality mapping in uniformly convex Banach space, which has proved to be a useful result [Ben-2], [Web-4], [Web-5]. Later we will use it to obtain a theorem on "dual" weak convergence for fixed point.

<u>Lemma 1.2.3.</u> (Webb, [Web-5]) *Let X be a separable, uniformly convex, Banach space with uniformly convex dual X^*. Let $\{x_n\}$ be a bounded sequence in X, then there exists a subsequence $\{x_k\}$ (say) such that $\phi(z)=\lim_{k\to\infty}\|x_k-z\|$ exists for all*

$z \in X$. *Moreover, there is a unique* $v \in X$ *such that* $w\text{-}\lim_{k\to\infty} F(x_k - v) = 0$ in X^*. *In fact* v *is the unique point in* X *at which* ϕ *attains its infimum.*

Remark. We suppose X is uniformly convex since we need the uniqueness of v. If one does not demand v unique, the uniformly convexity is not needed [Web-4].

Definition 1.2.4. The *characteristic of convexity* of a Banach space X is the number $\varepsilon_0(X) = \sup\{\varepsilon \geq 0 : \delta_X(\varepsilon) = 0\}$.

Next we state a useful Lemma on $\delta_X(\varepsilon)$, which can be found in [GoeK-1] (see Chapter 5).

Lemma 1.2.5. $\delta_X(\varepsilon)$ *is continuous on* $[0, 2)$ *and strictly increasing on* $[\varepsilon_0, 2]$. *Furthermove* $\delta_X(2-) = \lim_{\varepsilon\to 2-} \delta_X(\varepsilon) = 1 - \varepsilon_0(X)/2$.

Now we introduce a useful notion in fixed point theory, see [Ki].

Definition 1.2.6. A Banach space X is said to have *normal structure* if for every closed, bounded and convex subset C of X with $\operatorname{diam}(C) > 0$, $r(C, C) < \operatorname{diam}(C)$, where $r(C, C) = \inf_{x\in C}\left\{\sup_{y\in C}\|x-y\|\right\}$ is the Chebyshev radius of C.

Uniformly convex and uniformly smooth Banach spaces furnish examples of such spaces. If a Banach space X is uniformly convex or more generally has normal structure, a remarkable fixed point theorem for nonexpansive operators was discovered almost simultaneously by Browder [Bro], Gohde [Goh] and Kirk [Ki]. Recall that a Banach space X is said to have the f.p.p. (fixed point property), if for any nonempty, closed, bounded and convex subset C of X, every nonexpansive mapping $f: C \to C$ has a fixed point. Here f is nonexpansive means that $\|f(x)-f(y)\| \leq \|x-y\|$ for any $x, y \in C$. The fixed point theorem of Kirk is: If X is a reflexive Banach space with normal structure, then X has the f.p.p.. In chapter 3, we will give other characterizations of normal structure.

To obtain fixed point theorem in more general spaces, Bynum [By-2] defined three geometrical coefficients $N(X)$, $BS(X)$ and $WCS(X)$. Lim [Lim-2] proved $N(X)=BS(X)$ for any Banach space X. Here we give the definitions of $N(X)$ and $WCS(X)$ since we will study them and use them to study various measures of noncompactness in Chapters 2 and 3.

Definition 1.2.7. The *normal structure coefficient* $N(X)$ of a Banach space X is the number:

$$N(X)=\inf\Big\{\operatorname{diam}(C)/r(C, C) : C \text{ a closed, bounded, convex subset of } X \text{ with } \operatorname{diam}(C)>0\Big\}.$$

Definition 1.2.8. Let X be an infinite-dimensional reflexive Banach space, the *weakly convergent sequence coefficient* $WCS(X)$ of X is defined as:

$$WCS(X)=\inf\Big\{\operatorname{diam}_a\{x_n\}/\inf_{z\in\overline{\mathrm{co}}\{x_n\}}\phi(z) : \{x_n\}\subseteq X \text{ weakly convergent but not strongly convergent}\Big\}.$$

where $\phi(z)=\limsup_{n\to\infty}\|x_n-z\|$, $\operatorname{diam}_a\{x_n\}=\limsup_{n\to\infty}\Big\{\|x_m-x_k\| : m\geq n,\ k\geq n\Big\}$ is the asymptotic diameter of $\{x_n\}$.

Maluta [Mal] defined the following coefficient $D(X)$ which is related to $WCS(X)$ and to $N(X)$.

Definition 1.2.9. For a Banach space X, $D(X)$ is defined as:

$$D(X)=\sup\Big\{\limsup_{n\to\infty} \operatorname{dist}(x_{n+1}, \operatorname{co}\{x_i\}_1^n)/\operatorname{diam}\{x_n\} : \{x_n\}\subseteq X \text{ is bounded and nonconstant}\Big\}.$$

Obviously, both $N(X)$ and $WCS(X)$ are in $[1, 2]$. If F is a closed subspace of X it is easy to verify that $WCS(X)\leq WCS(F)$. Maluta showed that $D(X)\leq 1/N(X)$ and $D(X)=1$ if X is nonreflexive. Moreover, the numbers have the following relations: $N(X)\geq 1/(1-\delta_X(1))$ [By-2]; $D(X)\geq 1/2(1-\delta_X(1))$ [Mal]; if X is reflexive, $N(X)\leq WCS(X)$ [By-2]. Very recently, Benavides [Ben-3] and Prus [Pr] have proved $WCS(X)=1/D(X)$ for any infinite-dimensional reflexive Banach space X. We will study these coefficients in Chapters 2 and 3.

Next we give the definitions of other geometrical coefficients related to $N(X)$ and $WCS(X)$, which will be used in Chapters 2 and 4.

Definition 1.2.10. Let X be a Banach space. A related coefficient $\overline{N}(X)$ to the usual normal structure coefficient $N(X)$ is defined as:

$$\overline{N}(X)=\inf\Big\{\operatorname{diam}(C)/r(C, X) : C \text{ a closed, bounded, convex subset of } X \text{ with } \operatorname{diam}(C)>0\Big\},$$

where $r(C, X)=\inf_{x\in X}\Big\{\sup_{y\in C}\|x-y\|\Big\}$. Note that $2/\overline{N}(X)$ is also called Jung's constant (cf. [Am]).

Remark. Using a classical result of Klee and Garkavi (cf. [Ho], p.190), in any Hilbert space H, $r(C, X)=r(C, C)$ for C as above, so $N(X)=\overline{N}(X)$.

As a modification of $WCS(X)$, another number $W(X)$ is defined in [WebZ-2].

Definition 1.2.11. Let X be a reflexive Banach space, a geometrical coefficient $W(X)$ is defined as:

$$W(X)=\inf\left\{\operatorname{diam}\{x_n\}/\inf_{z\in X}\ \phi(z):\ \{x_n\}\subseteq X \text{ weakly convergent but not strongly convergent}\right\},$$

where $\phi(z)=\limsup_{n\to\infty}\|x_n-z\|$.

In certain spaces, the values of some numbers can be estimated. These estimates are useful in comparing certain geometrical coefficients and in the study of measures of noncompactness (Chapters 2, 3 and 4). Next we state several estimates for some numbers.

Theorem 1.2.12. (Jung, [Ju]) *For n-dimensional Euclidean space* l_n^2 $(n\in\mathbb{N})$, $\overline{N}(l_n^2)=(2(n+1)/n)^{1/2}$. *Let* E_n *be any n-dimensional Banach space, then* $\overline{N}(E_n)=2$ *if and only if* $E_n=l_n^\infty$.

Theorem 1.2.13. (Bohnenblust, [Bo]) *If* E_n *is a n-dimensional Banach space* $(n\in\mathbb{N})$, *then* $\overline{N}(E_n)\geq(n+1)/n$.

Theorem 1.2.14. (Maluta, [Mal]) *For n-dimensional Euclidean space* l_n^2 $(n\in\mathbb{N})$, $N(l_n^2)=(2(n+1)/n)^{1/2}$. *In every infinite dimensional Hilbert space H,* $WCS(H)=\sqrt{2}$, *and* $N(l^2)=\sqrt{2}$.

Theorem 1.2.15. (Maluta, [Mal]) *In every infinite-dimensional Banach space,* $N(X)\leq\sqrt{2}$.

Theorem 1.2.16. (Bynum, [By-2]) *For* $1<p<\infty$, $WCS(l^p)=2^{1/p}$.

The next result was proved independently by Benavides and Prus at about the same time, it can be found in [Ben-3] or [Pr].

Theorem 1.2.17. *For* $1<p<\infty$, $N(l^p)=N(L^p)=WCS(L^p)=\min\{2^{1/p}, 2^{1-1/p}\}$.

Remark. l^p $(1<p<2)$ spaces give examples where $N(X)<WCS(X)$.

The existence of fixed points for uniformly Lipschitzian mapping has been well studied in recent years, e.g. [GoeK-2], [GoeKT], [Lif], [Lim-1], [CaM]. To obtain theorems of this type, certain geometrical ceofficients are helpful. Here we introduce one of these geometrical coefficients — the Lifschitz characteristic defined by Lifschitz [Lif] in order to obtain a fixed point theorem. We will study this number in Chapter 2.

Definition 1.2.18. Let (M, ρ) be a metric space, the *Lifschitz characteristic* $\kappa(M)$ of M is defined by

$$\kappa(M)=\sup\{b>0 : \text{there exists } a>1 \text{ such that for all } x, y\in M \text{ and all } r>0,\ \rho(x, y)>r \text{ implies there exists } z\in M \text{ such that } \overline{B}(x, br)\cap\overline{B}(y, ar)\subseteq\overline{B}(z, r)\}.$$

Obviously $1\leq\kappa(M)\leq 2$. If M is a nonreflexive Banach space, then for any closed ball B in M, $\kappa(B)=1$ (cf. [KrZ], p.227). If H is a Hilbert space, $\kappa(H)\geq\sqrt{2}$ ([Lif] or [KrZ], p.227), see also Theorem 2.1.7. It is easy to see that for a Banach space X,

$$\kappa(X)=\sup\{b>0 : \text{there exists } a>1 \text{ such that for all } y\in X \text{ with } \|y\|>1, \text{ there exists } z\in X \text{ such that } \overline{B}(0, b)\cap\overline{B}(y, a)\subseteq\overline{B}(z, 1)\}.$$

One reason for studying $\kappa(M)$ is the following fixed point theorem of Lifschitz, see [Lif] or [KrZ], p.227, Theorem 37.11.

Theorem 1.2.19. *Let* (M, ρ) *be a complete metric space and let* f *be a mapping* *from M into itself. Assume that*

1) *f is uniformly k-lipschitzian for some $k<\kappa(M)$, that is, for any $x, y \in M$ and $n \in \mathbb{N}$, $\rho(f^n x, f^n y) \le k\rho(x, y)$;*

2) *there is $x_0 \in M$ so that $\{f^n x_0\}_1^\infty$ is bounded.*

Then f has at least one fixed point in M.

<u>Definition 1.2.20.</u> (cf. [DowT]) Let X be a Banach space, $\kappa_0(X)$ is defined by

$$\kappa_0(X)=\inf\{\kappa(C) : C \text{ a closed, bounded, convex subset of } X \text{ with } \operatorname{diam}(C)>0\}.$$

Let X be a Banach space and let C be a closed, bounded and convex set in X. If $f: C \longrightarrow C$ is uniformly k-lipschitzian for some $k<\kappa_0(X)$, from Theorem 1.2.19, we see that f has fixed points in C.

In Chapter 2, we will estimate the values of $\kappa(X)$ and $\kappa_0(X)$ for some spaces. So fixed point theorems in certain spaces can be obtained as consequence of Theorem 1.2.19 .

The following result is shown in [DowT] and relates $\kappa_0(X)$ and $\varepsilon_0(X)$.

<u>Theorem 1.2.21.</u> (Downing and Turett) *Let X be a Banach space. If $\gamma>1$ satisfies $\gamma(1-\delta_X(1/\gamma))=1$, then $\gamma \le \kappa_0(X)$. Also $\varepsilon_0(X)<1$ if and only if $\kappa_0(X)>1$.*

<u>1.3. Measures of noncompactness</u>

The notion of the measure of noncompactness is of importance in the study of classes of noncompact operators, see, for example [De], [EdE], [Ll], [Mas]. It has also proved useful in the study of reflexivity and other geometrical properties of Banach space [Ban], [BenL-2], [GoeS], [Mo], [Ro-1] and [Ro-2]. In this section, we introduce three kinds of measures of noncompactness, which have good properties and have been well studied in recent years.

Let M be a metric space, $\mathscr{B}$ the collection of bounded subsets of M. A measure of noncompactness on M is a function $\mu : \mathscr{B} \longrightarrow [0, +\infty)$ with the properties that:

1) $\mu(\Omega)=0$ if and only if Ω is precompact, and

2) $\mu(\Omega_1)\leq\mu(\Omega_2)$ if $\Omega_1\subseteq\Omega_2$.

Next, we give the definitions of three well-known measures of noncompactness.

Definition 1.3.1. Let Ω be a bounded subset of a metric space (M, ρ). The *set (or Kuratowski) measure of noncompactness* $\alpha(\Omega)$ of Ω is defined by

$$\alpha(\Omega)=\inf\left\{\varepsilon>0 : \Omega \text{ can be covered by a finite number of sets of diameter} \leq\varepsilon\right\};$$

the *ball measure of noncompactness* $\beta(\Omega)$ of Ω is defined by

$$\beta(\Omega)=\inf\left\{\varepsilon>0 : \Omega \text{ can be covered by a finite number of balls of diameter} \leq\varepsilon\right\};$$

the *separation measure of noncompactness* $\delta(\Omega)$ of Ω is defined by

$$\delta(\Omega)=\sup\left\{\varepsilon\geq 0 : \text{there is a sequence } \{x_n\}\subseteq\Omega \text{ such that } \rho(x_n, x_m)\geq\varepsilon \text{ if } m\neq n\right\}.$$

The set and ball measures have been particularly well studied and can be found in many books on nonlinear functional analysis, e.g. [De], [EdE], [Ll]. It is easy to see that $\alpha(\Omega)\leq\beta(\Omega)\leq 2\alpha(\Omega)$ and these inequalities are best possible in general. To illustrate this, take any infinite dimensional Banach space X, it is known that $\alpha(\overline{B}_X)=\beta(\overline{B}_X)=2$ (Nussbaum, cf. [Ll], p.93). To get a set Ω such that $\beta(\Omega)=2\alpha(\Omega)$, we consider the space (c_0) of sequence that converges to 0 with the maximum norm. Let $e_n=(\delta_{nm})$, where $\delta_{nm}=0$ if $m\neq n$; $\delta_{nm}=1$

if $m=n$. Then $\alpha(\{e_n\})=1$ since the diameter of any subsequence of $\{e_n\}$ is 1. Clearly $\beta(\{e_n\})\leq 2$. If $\beta(\{e_n\})<2$, then for any $a\in\mathbb{R}$ such that $\beta(\{e_n\})/2<a<1$, there are $z_1,\ldots,z_p$ so that $\{e_n\}\subseteq\bigcup_{i=1}^{p}B(z_i,a)$. Some $B(z_j,a)$ $(1\leq j\leq p)$ contains a subsequence $\{e_{n_k}\}$ of $\{e_n\}$, that is, $\|e_{n_k}-z_j\|\leq a$. Let $z_j=(z_{jm})$, then $\|e_{n_k}-z_j\|\geq|1-z_{jn_k}|\longrightarrow 1$ $(k\longrightarrow\infty)$, a contradiction. Hence $\beta(\{e_n\})=2\alpha(\{e_n\})=2$.

The separation measure of noncompactness may be found in [WelW], p.90. The relationship between $\delta(\Omega)$, $\alpha(\Omega)$ and $\beta(\Omega)$ is $\delta(\Omega)\leq\alpha(\Omega)\leq\beta(\Omega)\leq 2\delta(\Omega)$. Consider the above example in (c_0), $\delta(\{e_n\})=1$. This shows that the inequalities just mentioned are best possible in general. For an infinite-dimensional Banach space X, unlike the set and ball measures, $\delta(\overline{B}_X)$ is not always 2, but $\delta(\overline{B}_X)>1$ is always true [ElO]. In many spaces, $\delta(\overline{B}_X)<2$. The next theorem shows the exact values of $\delta(l^p)$ and $\delta(L^p)$ (see [WelW], p.91, Theorem 16.9), which are useful to relate $\delta(\Omega)$ and $\beta(\Omega)$ (see Chapter 4).

<u>Theorem 1.3.2.</u> *For $1\leq p\leq\infty$, let $\overline{B}_p$ denote the closed unit ball of l^p, $\overline{B}^p$ the closed unit ball of L^p. Then for $1\leq p<\infty$, $\delta(\overline{B}_p)=2^{1/p}$ and $\delta(\overline{B}^p)=\max\{2^{1/p},2^{1-1/p}\}$; $\delta(\overline{B}^{\infty})=\delta(\overline{B}_{\infty})=2$.*

Although the three measures are not equal, they share many similar properties. Next we state some of the more important of these. We will use these properties in the proofs of some results in Chapters 3, 4 and 5.

<u>Lemma 1.3.3.</u> *Let Ω_i, $i=1,2$, be bounded subsets of a complete metric space M. We use ψ to denote any of the three measures α, β, or δ, then:*

1) $\psi(\Omega_1)=0$ *if, and only if, Ω_1 is precompact;*

2) *if $\Omega_1\subseteq\Omega_2$, then $\psi(\Omega_1)\leq\psi(\Omega_2)$;*

3) $\psi(\overline{\Omega}_1)=\psi(\Omega_1)$;

4) $\psi(\Omega_1\cup\Omega_2)=\max\{\psi(\Omega_1),\ \psi(\Omega_2)\}$.

All the above properties are easy to verify (see [EdE], p.13). Next we give more properties of these measures of noncompactness for Banach spaces (cf. [EdE], p.15).

Lemma 1.3.4. *Let X be a Banach space, and let Ω_i, $i=1,2$, be bounded subsets of X. We use γ to denote α or β, then we have*

5) $\gamma(\Omega_1+\Omega_2)\leq\gamma(\Omega_1)+\gamma(\Omega_2)$;

6) $\gamma(\text{co}\Omega_1)=\gamma(\Omega_1)$;

7) $\gamma(a\Omega_1)=|a|\gamma(\Omega_1)$ *for any* $a\in\mathbb{R}$.

The result corresponding to 6) for the measure δ is surprisingly difficult to prove. $\alpha(\text{co}\Omega)=\alpha(\Omega)$ and $\beta(\text{co}\Omega)=\beta(\Omega)$ are fundamental results, but $\delta(\text{co}\Omega)=\delta(\Omega)$ is obtained very recently by Arias [Ari], and the proof is quite complicated. The property that their values at Ω and at coΩ are the same implies that some contractive type mappings have fixed points [Mas], so is a useful one.

Lemma 1.3.5. (Arias, [Ari]) *Let Ω be a bounded subset of Banach space X, then*

8) $\delta(\text{co}\Omega)=\delta(\Omega)$.

Next we introduce the notions of α-minimal and β-minimal sets defined by Benavides [Ben-1] and give some properties of them. These definitions and properties have proved very useful in the study of various measures of noncompactness, e.g. [Ben-1], [Ben-2], [BenA], [BenL-1], [BenL-2], [WebZ-2]. We will use these notions and properties frequently in Chapters 3 and 4.

Definition 1.3.6. An infinite set Ω in a metric space M is said to be *α-minimal* (*β-minimal*) if $\alpha(\Omega)=\alpha(D)$ ($\beta(\Omega)=\beta(D)$) for any infinite subset D of Ω.

Note that every infinite subset of an α-minimal (β-minimal) set is α-minimal (β-minimal); any infinite precompact set is α-minimal and β-minimal.

Proposition 1.3.7. (Benavides, [Ben-1]) *Let M be a bounded metric space with infinitely many points. Then there exists an α-minimal subset Ω of M. Furthermore if M is not a precompact set, Ω can be chosen such that $\alpha(\Omega)>0$.*

The above result is also true for β-minimal sets. When M is separable, an improved result for β-minimal sets can be obtained.

Proposition 1.3.8. (Benavides, [Ben-1]) *Let M be a separable metric space and Ω an infinite bounded subset of M. Then there is a β-minimal subset D of Ω such that $\beta(D)=\beta(\Omega)$.*

The corresponding Proposition for the α measure does not hold. An example is the unit sphere S_H in infinite-dimensional Hilbert space H. $\alpha(S_H)=2$, but for any α-minimal subset A of S_H, $\alpha(A)\leq\sqrt{2}$ (see [Ben-1]). Later, we will see several other examples in Chapter 4.

The next result is also due to Benavides [Ben-1], but we give a more elementary proof.

Lemma 1.3.9. *Let Ω be an α-minimal subset of a metric space (M,ρ). Then for every $\varepsilon>0$, there exists an infinite subset D of Ω such that*

$$\alpha(\Omega)-\varepsilon<\rho(x,y)<\alpha(\Omega)+\varepsilon \quad \textit{for every } x,y\in D.$$

Proof. Let Ω_1 be any infinite subset of Ω. For any $\varepsilon>0$, we claim that there are $x_1\in\Omega_1$ and an infinite subset Ω_2 of Ω_1 such that $\rho(x_1,z)>\alpha(\Omega)-\varepsilon$ for every $z\in\Omega_2$. Otherwise, let x_1 be any point in Ω_1, there is $x_2\in\Omega_1$, $x_2\neq x_1$ such that $\rho(x_1,x_2)\leq\alpha(\Omega)-\varepsilon$. When $x_1,\ldots,x_n$ are given, since

$$G:=\bigcup_{i=1}^{n}\left\{z\in\Omega_1:\ \rho(x_i,z)>\alpha(\Omega)-\varepsilon\right\}$$

is finite, $\Omega_1\backslash G$ is an infinite set. Let $x_{n+1}\in\Omega_1\backslash G$, with $x_{n+1}\neq x_i$, then $\rho(x_{n+1},x_i)\leq\alpha(\Omega)-\varepsilon$ for all $i=1,\ldots,n$. As a result, we obtain a sequence $\{x_n\}$ in Ω with $x_n\neq x_m$ if $n\neq m$ and $\mathrm{diam}(\{x_n\})\leq\alpha(\Omega)-\varepsilon$, this contradicts the α-minimality of Ω.

For any $\varepsilon>0$, there are $\Omega_i'\subseteq\Omega$, $i=1,\ldots,n$, so that $\Omega=\bigcup_{i=1}^{n}\Omega_i'$ and $\mathrm{diam}(\Omega_i')<\alpha(\Omega)+\varepsilon$. There is at least one Ω_i' which is an infinite set. Denote this set by D_1. Then $\alpha(D_1)=\alpha(\Omega)$ and $\rho(x,y)<\alpha(\Omega)+\varepsilon$ for every $x,y\in D_1$. There are $x_1\in D_1$ and an infinite subset D_2 of D_1 so that $\rho(x_1,z)>\alpha(\Omega)-\varepsilon$ for every $z\in D_2$. For D_2, there are $x_2\in D_2$ and an infinite subset D_3 of D_2 so that $\rho(x_2,z)>\alpha(\Omega)-\varepsilon$ for every $z\in D_3$. Continue the process, we obtain sequences $\{x_n\}$ and $\{D_n\}$ satisfying:

1) $x_n\in D_n$;

2) $D_{n+1}\subseteq D_n$ and every D_n is infinite set;

3) $\rho(x_n,z)>\alpha(\Omega)-\varepsilon$ for every $z\in D_{n+1}$.

Hence we have $\rho(x_i,x_j)>\alpha(\Omega)-\varepsilon$ if $i\neq j$. $D:=\{x_n\}$ satisfies the conclusion of the Lemma.

Let M and E be metric spaces, the mapping $f:M\longrightarrow E$ is called *compact* if it is continuous and for every bounded subset Ω of M, $\overline{f(\Omega)}$ is compact. The measures of noncompactness enable us to define several classes of noncompact and contractive type mappings which are useful in the study of nonlinear operator theory.

Definition 1.3.10. Let M and E be two metric spaces and let $k \geq 0$. A mapping $f : M \longrightarrow E$ is called a *k-set contraction* if, and only if, it is continuous and for every bounded set Ω in M, we have $\alpha(f(\Omega)) \leq k\alpha(\Omega)$; it is called a *k-ball contraction* if, and only if, it is continuous and $\beta(f(\Omega)) \leq k\beta(\Omega)$ for every bounded set Ω in M; it is called a *k-δ-contraction* if, and only if, it is continuous and $\delta(f(\Omega)) \leq k\delta(\Omega)$ for every bounded set $\Omega \subseteq M$.

Contractions, compact mappings and sums of these furnish examples of such mappings. For a k-set contraction f (it is also a k-ball contraction and a k-δ-contraction for possible different values of k), the measures of noncompactness for operators, $\alpha(f)$, $\beta(f)$, and $\delta(f)$, can be defined.

Definition 1.3.11. If $f : M \longrightarrow E$ is a k-set contraction for some $k \geq 0$, define

$$\alpha(f) = \inf\{k \geq 0 : f \text{ is a } k\text{-set contraction}\};$$

$$\beta(f) = \inf\{k \geq 0 : f \text{ is a } k\text{-ball contraction}\};$$

$$\delta(f) = \inf\{k \geq 0 : f \text{ is a } k\text{-}\delta\text{-contraction}\}.$$

If f is not a k-set contraction for any $k \geq 0$, we set $\alpha(f) = \beta(f) = \delta(f) = +\infty$. Then $\alpha(f)$, $\beta(f)$ and $\delta(f)$ are defined for all continuous mappings. Obviously, f is compact if and only if $\alpha(f) = \beta(f) = \delta(f) = 0$. Generally, $\beta(f)/2 \leq \alpha(f) \leq 2\beta(f)$ are the best possible relationships between $\alpha(f)$ and $\beta(f)$; and $\beta(f)/2 \leq \delta(f) \leq 2\beta(f)$ are the best possible inequalities between $\delta(f)$ and $\beta(f)$. Also we have $\alpha(f)/2 \leq \delta(f) \leq 2\alpha(f)$. Later we will show that $\delta(f) \leq \alpha(f)$ and improve on $\beta(f)/2 \leq \alpha(f) \leq 2\beta(f)$ and $\beta(f)/2 \leq \delta(f) \leq 2\beta(f)$.

If X and Y are two Banach spaces, let $\mathscr{K}(X, Y)$ be the subspace of

$\mathscr{L}(X, Y)$ consisting of the compact operators. For $T \in \mathscr{L}(X, Y)$, the $\mathscr{K}$-seminorm of T is defined as $\|T\|_{\mathscr{K}} = \inf\{\|T+C\|: C \in \mathscr{K}(X, Y)\}$. It is easy to see that T is $\|T\|_{\mathscr{K}}$-set contraction, $\|T\|_{\mathscr{K}}$-ball contraction, and $\|T\|_{\mathscr{K}}$-δ-contraction. In order to study noncompact operators in $\mathscr{L}(X, Y)$, investigations on seminorms on $\mathscr{L}(X, Y)$ are helpful. It is easy to verify that $\alpha(T)$, $\beta(T)$ and $\delta(T)$ are all seminorms. Also $\beta(T) = \beta(T(\overline{B}_X))/2$ (cf. [Web-2]). Some relationships between certain seminorms of T and T^* are well studied, we state several of these results next. In Chapter 5, we will use these results and give more results on seminorms.

Theorem 1.3.12. (Nussbaum, [Nu-1]) *Let X and Y be Banach spaces and let $T \in \mathscr{L}(X, Y)$. Then $\alpha(T) \leq \beta(T^*)$ and $\alpha(T^*) \leq \beta(T)$.*

Corollary 1.3.13. (Goldenstein and Markus, cf. [EdE], p.19) *Let X and Y be Banach spaces and let $T \in \mathscr{L}(X, Y)$. Then*

$$\alpha(T)/2 \leq \alpha(T^*) \leq 2\alpha(T) \text{ and } \beta(T)/2 \leq \beta(T^*) \leq 2\beta(T).$$

Theorem 1.3.14. (Webb, [Web-1] and [Web-2]) *Let H and H_1 be Hilbert spaces and let $T \in \mathscr{L}(H, H_1)$. Then $\beta(T^*) \leq \beta(T)$ and $\alpha(T) = \beta(T)$.*

Remark. From Theorems 1.3.12 and 1.3.14, we have $\beta(T^*) = \beta(T)$ for $T \in \mathscr{L}(H, H_1)$.

CHAPTER TWO

THE LIFSCHITZ CHARACTERISTIC

The Lifschitz characteristic has proved to be a very useful geometrical notion in fixed point theory. However, what the relationships between it and other geometrical coefficients are, and how its values in classical spaces are calculated, are seldom studied. In this chapter, we will connect the Lifschitz characteristic and other numbers defined in chapter one and estimate its values in certain spaces. Also some fixed point theorems which employ the Lifschitz characteristic are given.

This chapter includes the work of [WebZ-1] and part of the work of [WebZ-2].

2.1. Some properties of the Lifschitz characteristic

We will give some properties of the Lifschitz characteristic and will connect it with other geometrical coefficients. Also some examples are given to shown the difference between it and other numbers.

The next Lemma is taken from [WebZ-2] and is the essential one needed for relating the Lifschitz characteristic and other numbers.

Lemma 2.1.1. *Let (M, p) be a metric space and let Ω be a bounded subset of M with diameter d. Then for any $0<b<\kappa(M)$, there exists $z\in M$ such that $\Omega\subseteq\overline{B}(z, d/b)$.*

Proof. If $b \le 1$, then for any $z \in \Omega$, $\Omega \subseteq \overline{B}(z, d) \subseteq \overline{B}(z, d/b)$. Now suppose that $\kappa(M) > 1$ and that $1 < b < \kappa(M)$. By the definition of $\kappa(M)$, there is $a > 1$ such that for all $x, y \in M$ and all $r > 0$, $\rho(x, y) > r$ implies that there exists $z \in M$ such that $\overline{B}(y, ar) \cap \overline{B}(x, br) \subseteq \overline{B}(z, r)$.

Case 1 : $a \ge b$. Take $r = d/b < d$. Then there are points $x, y \in \Omega$ with $\rho(x, y) > r$. Hence, there exists $z \in M$ such that $\overline{B}(y, ad/b) \cap \overline{B}(x, d) \subseteq \overline{B}(z, d/b)$. For any $w \in \Omega$, $\rho(w, y) \le d \le ad/b$ and $\rho(w, x) \le d$, so $w \in \overline{B}(y, ad/b) \cap \overline{B}(x, d)$. Therefore, $w \in \overline{B}(z, d/b)$, and hence $\Omega \subseteq \overline{B}(z, d/b)$ since w is arbitrary.

Case 2 : $a < b$. Since $a > 1$, there is an integer $N \ge 1$ such that $a^N < b \le a^{N+1}$. Let $\gamma \le 1$ be so that $\gamma a^{N+1} = b$; note that $\gamma a > 1$. For $n = 1, 2, \ldots, N+1$, let $r_n = d/(\gamma a^n)$. We claim that there exists $z_n \in M$ such that $\Omega \subseteq \overline{B}(z_n, r_n)$ for all n: $1 \le n \le N+1$. Indeed, for $n = 1$, $r_1 < d$, so there are $x, y \in \Omega$ with $\rho(x, y) > r_1$, which gives the existence of $z_1 \in M$ such that

$$\overline{B}(y, ar_1) \cap \overline{B}(x, br_1) \subseteq \overline{B}(z_1, r_1).$$

Since $br_1 > ar_1 = d/\gamma \ge d$, we have $\rho(w, y) \le d \le ar_1$ and $\rho(w, x) \le d \le br_1$ for any $w \in \Omega$. Hence the left hand side contains w, so it contains Ω since w is arbitrary. Therefore $\Omega \subseteq \overline{B}(z_1, r_1)$. This proves the case $n = 1$. Now suppose the above has been shown for $n = i \le N$, i.e., there exists $z_i \in M$ such that $\Omega \subseteq \overline{B}(z_i, r_i)$. If $\rho(x, z_i) \le r_{i+1}$ for all $x \in \Omega$, then we may take $z_{i+1} = z_i$, $\Omega \subseteq \overline{B}(z_{i+1}, r_{i+1})$. Otherwise, there exists $x \in \Omega$ such that $\rho(x, z_i) > r_{i+1}$. This yields $z_{i+1} \in M$ such that

$$\overline{B}(z_i, ar_{i+1}) \cap \overline{B}(x, br_{i+1}) \subseteq \overline{B}(z_{i+1}, r_{i+1}).$$

For $u \in \Omega$, we have $\rho(u, z_i) \le r_i = ar_{i+1}$ and $\rho(u, x) \le d = \gamma a^{i+1} r_{i+1} \le br_{i+1}$, since $\gamma a^{i+1} \le \gamma a^{N+1} = b$. So $u \in \overline{B}(z_i, ar_{i+1}) \cap \overline{B}(x, br_{i+1})$. Therefore $u \in \overline{B}(z_{i+1}, r_{i+1})$. By the arbitrariness of $u \in \Omega$, we have $\Omega \subseteq \overline{B}(z_{i+1}, r_{i+1})$. Hence, in finitely many steps, we obtain the claimed result. Since $r_{N+1} = d/b$, the proof of the Lemma is complete.

Now we can connect $\kappa_0(X)$ and $N(X)$, the result mentioned in [WebZ-2], which does not seem to have been noted previously.

Theorem 2.1.2. *Let X be a Banach space, then $\kappa_0(X) \leq N(X)$.*

Proof. Let C be any closed, bounded and convex set in X with $\mathrm{diam}(C) > 0$, then from Lemma 2.1.1, for any $b: 0 < b < \kappa(C)$, there is $z \in C$ such that $C \subseteq \overline{\mathrm{B}}(z, \mathrm{diam}(C)/b)$. Thus $r(C, C) \leq \sup_{x \in C} \|x - z\| \leq \mathrm{diam}(C)/b$. Then $b \leq \mathrm{diam}(C)/r(C, C)$. Since b is arbitrary, we have $\kappa(C) \leq \mathrm{diam}(C)/r(C, C)$. By the definitions of $\kappa_0(X)$ and $N(X)$, we obtain $\kappa_0(X) \leq N(X)$.

Remark. If X is reflexive, by Theorem 2.1.2 and results in section 2 in chapter 1, we obtain the following inequalities: $1 \leq \kappa_0(X) \leq N(X) \leq WCS(X) \leq 2$.

By Theorems 1.2.15 and 2.1.2, we have the following result.

Corollary 2.1.3. *If X is an infinite dimensional Banach space, then $\kappa_0(X) \leq \sqrt{2}$.*

The next lemma is given in [Lif], the corresponding results for $\kappa(X)$ is a trivial consequence of the definition.

Lemma 2.1.4. (Lifschitz) *Let X be a Banach space, then*

$$\kappa_0(X) \geq \sup\{b > 0 : \text{there exists } a > 1 \text{ such that for all } y \in X \text{ with } \|y\| > 1, \\ \text{there exists } t \in [0, 1] \text{ such that } \overline{\mathrm{B}}(0, b) \cap \overline{\mathrm{B}}(y, a) \subseteq \overline{\mathrm{B}}(ty, 1)\}.$$

Proof. We give the proof here since the paper of Lifschitz is only available in Russian. We denote the set on the right hand side as A. For any $b \in A$, there exists $a > 1$ such that for all $y \in X$ with $\|y\| > 1$, there exists $t \in [0, 1]$ such that

$\overline{B}(0, b)\cap\overline{B}(y, a)\subseteq\overline{B}(ty, 1)$. Let C be a closed, bounded, convex subset of X with $\operatorname{diam}(C)>0$. For any $x, y\in C$ and $r>0$, if $\|x-y\|>r$, we have $\|(x-y)/r\|>1$. Hence there is $t'\in[0, 1]$ such that $\overline{B}(0, b)\cap\overline{B}((y-x)/r, a)\subseteq\overline{B}(t'(y-x)/r, 1)$. Then

$$\begin{aligned}\overline{B}(x, br)\cap\overline{B}(y, ar)&=x + \overline{B}(0, br)\cap\overline{B}(y-x, ar)\\ &=x + r\left[\overline{B}(0, b)\cap\overline{B}((y-x)/r, a)\right]\\ &\subseteq x + r\,\overline{B}(t'(y-x)/r, 1)=\overline{B}(x+t'(y-x), r).\end{aligned}$$

Since C is convex, $x+t'(y-x)\in C$. By the definition of $\kappa(C)$, $\kappa(C)\geq b$. By the definition of $\kappa_0(X)$, $\kappa_0(X)\geq b$. Hence $\kappa_0(X)\geq\sup A$.

Next we give a lower bound for $\kappa_0(X)$ by using the modulus of convexity $\delta_X(\varepsilon)$.

Theorem 2.1.5. *Let X be a Banach space, then $\kappa_0(X)\geq 1/(1-\delta_X(1))$.*

Proof. If $\delta_X(1)=0$, there is nothing to prove. Now suppose $\delta_X(1)>0$ and let $1<b<1/(1-\delta_X(1))$. Since $\delta_X(\varepsilon)$ is continuous on $[0, 2)$, there is $c\in(1/b, 1)$ such that $b<1/(1-\delta_X(c))$. Let $a=\min\{bc, 1+b(1-c)\}>1$. We claim that for any $y\in X$ with $\|y\|>1$, there is $t\in[0, 1]$ such that $\overline{B}(0, b)\cap\overline{B}(y, a)\subseteq\overline{B}(ty, 1)$.

If $\|y\|\geq b$, then for any $x\in\overline{B}(0, b)\cap\overline{B}(y, a)$, we have $\|(x-y)/b\|\leq a/b\leq c<1$, $\|x/b\|\leq 1$, and $\|x/b - (x-y)/b\|=\|y/b\|\geq 1$. Hence

$$\delta_X(1)\leq 1-\|x/b+(x-y)/b\|/2=1-\|x-y/2\|/b,$$

that is $\|x-y/2\|\leq b(1-\delta_X(1))<1$. Therefore $\overline{B}(0, b)\cap\overline{B}(y, a)\subseteq\overline{B}(y/2, 1)$.

If $b>\|y\|\geq a$, then for any $x\in\overline{B}(0, b)\cap\overline{B}(y, a)$, we have $\|x/b\|\leq 1$ and

$$\begin{aligned}\|\, x/b - y/\|y\|\,\| &\leq \|x-y\|/b + |\, 1/b - 1/\|y\|\,|\;\|y\|\\ &\leq a/b + |\,\|y\|/b - 1\,| =a/b+1-\|y\|/b\leq 1,\end{aligned}$$

also we have $\|x/b-(x/b - y/\|y\|)\|=1$. Hence $\|x/b+(x/b - y/\|y\|)\|/2\leq 1-\delta_X(1)$, that is $\|x-[b/(2\|y\|)]y\|\leq b(1-\delta_X(1))<1$. Therefore $\overline{B}(0, b)\cap\overline{B}(y, a)\subseteq\overline{B}([b/(2\|y\|)]y, 1)$.

If $a > \|y\| > 1$, let $\lambda = c/\|y\|$, then for any $x \in \overline{B}(0, b) \cap \overline{B}(y, a)$, we have $\|x/b\| \le 1$ and

$$\|x/b - \lambda y\| \le \|x-y\|/b + |1/b - \lambda| \ \|y\| \le a/b + | \ \|y\|/b - c|$$
$$= a/b + c - \|y\|/b \le a/b + c - 1/b \le [1 + b(1-c)]/b + c - 1/b = 1,$$

also we have $\|x/b - (x/b - \lambda y)\| = \lambda \|y\| = c$. Hence $\|x/b + (x/b - \lambda y)\|/2 \le 1 - \delta_X(c)$, that is $\|x - (b\lambda/2)y\| \le b(1-\delta_X(c)) < 1$. Therefore $\overline{B}(0, b) \cap \overline{B}(y, a) \subseteq \overline{B}((b\lambda/2)y, 1)$.

By Lemma 2.1.4, we obtain $\kappa_0(X) \ge 1/(1-\delta_X(1))$.

Remark. The above result is mentioned in [WebZ-2] as a remark, but the simple reason given there is wrong. This result is better than $\kappa_0(X) \ge \gamma$ with $\gamma(1-\delta_X(1/\gamma)) = 1$ in Theorem 1.2.21. In fact, we have $1-\delta_X(1) \le 1-\delta_X(1/\gamma) = 1/\gamma$ since $\delta_X(\varepsilon)$ is increasing, so $1/(1-\delta_X(1)) \ge \gamma$.

Now we give connections between $\kappa(X)$ and other geometrical numbers.

Theorem 2.1.6. *If X is a Banach space, then $1 \le \kappa(X) \le \overline{N}(X) \le 2$. If X is reflexive, then $1 \le \kappa(X) \le \overline{N}(X) \le W(X) \le 2$.*

Proof. Let C be any closed, bounded and convex set in X with $\text{diam}(C) > 0$, then from Lemma 2.1.1, for any b: $0 < b < \kappa(X)$, there is $z \in X$ such that $C \subseteq \overline{B}(z, \text{diam}(C)/b)$. Thus $r(C, X) \le \sup_{x \in C} \|x - z\| \le \text{diam}(C)/b$. Then $b \le \text{diam}(C)/r(C, X)$. By the arbitrariness of b, we have $\kappa(X) \le \text{diam}(C)/r(C, X)$. From the definition of $\overline{N}(X)$, we obtain $\kappa(X) \le \overline{N}(X)$.

$W(X) \le 2$ follows from the triangle inequality. The only thing left needed to prove is $\overline{N}(X) \le W(X)$ for reflexive space X. Let $\{x_n\}$ be any weakly convergent but not strongly convergent sequence in X and let $C = \overline{\text{co}}\{x_n\}$. Then $\text{diam}(C) = \text{diam}\{x_n\}$. For any $z \in X$, $\|x_n - z\| \le \sup_{y \in C} \|y - z\|$, then $\sup_{y \in C} \|y - z\| \ge \phi(z)$, where $\phi(z) = \limsup_{n \to \infty} \|x_n - z\|$. Thus $r(C, X) = \inf_{z \in X} \sup_{y \in C} \|y - z\| \ge \inf_{z \in X} \phi(z)$. Therefore we obtain

$$\operatorname{diam}(\{x_n\})/\inf_{z\in X}\phi(z)\geq \operatorname{diam}(C)/r(C, X).$$

From the definitions of $W(X)$ and $\overline{N}(X)$, the conclusion is shown.

Next we give the relationship between $\kappa(X)$ and $\kappa_0(X)$, a result mentioned in [WebZ-2].

<u>Proposition 2.1.7.</u> *Let X be a Banach space, then* $\kappa(X)\geq\kappa\big(\overline{\mathrm{B}}(0, 4)\big)\geq\kappa_0(X)$.

<u>Proof.</u> For any b: $0<b<\kappa\big(\overline{\mathrm{B}}(0, 4)\big)$, there is an $a>1$ such that for any $x, y\in\overline{\mathrm{B}}(0, 4)$, and any $r>0$, $\|x-y\|>r$ implies there exists $z\in\overline{\mathrm{B}}(0, 4)$ such that

$$\overline{\mathrm{B}}(x, br)\cap\overline{\mathrm{B}}(y, ar)\subseteq\overline{\mathrm{B}}(z, r).$$

For a and b as above, if $y\in X$ is such that $1<\|y\|<4$, then $\overline{\mathrm{B}}(0, b)\cap\overline{\mathrm{B}}(y, a)\subseteq\overline{\mathrm{B}}(z, 1)$ for some $z\in\overline{\mathrm{B}}(0, 4)\subseteq X$. If $\|y\|\geq 4$, $\overline{\mathrm{B}}(0, b)\cap\overline{\mathrm{B}}(y, 3/2)=\emptyset$ since $b<2$. Thus, by the equivalent definition of $\kappa(X)$, $\kappa(X)\geq b$. Therefore $\kappa(X)\geq\kappa\big(\overline{\mathrm{B}}(0, 4)\big)$ as b is arbitrary.

<u>Remark.</u> We do not know whether $\kappa(X)=\kappa_0(X)$ is true or not. We think at least $\kappa(X)=\kappa\big(\overline{\mathrm{B}}(0, 4)\big)$ should be true.

Using Lemma 2.1.4, we can see that the real number set $\mathbb{R}$ gives a simple example of when the values of κ and κ_0 are equal: $\kappa(\mathbb{R})=\kappa_0(\mathbb{R})=2$. In fact, for $y\in\mathbb{R}$ and $a, b\in\mathbb{R}^+$ with $1<a\leq b$, if $|y|>1$, then

$$\begin{aligned}\overline{\mathrm{B}}(0, b)\cap\overline{\mathrm{B}}(y, a)=[-b, b]\cap[y-a, y+a]&=\emptyset && \text{if } y>a+b \text{ or } y<-(a+b)\\ &=[y-a, b] && \text{if } 1<y\leq a+b\\ &=[-b, y+a] && \text{if } -1>y\geq-(a+b).\end{aligned}$$

For any b : $1<b<2$, let $a=\min\{b, 3-b\}>1$. Then for $1<y\leq a+b$ or $-1>y\geq-(a+b)$, it is easy to see that

$$\overline{B}(0, b)\cap\overline{B}(y, a)\subseteq\overline{B}(ty, (b+a-|y|)/2),$$

where $t=(b-a+y)/(2y)$ for $1<y<a+b$, $t=(y+a-b)/(2y)$ for $-1>y\geq-(a+b)$. Since $(b+a-|y|)/2<(b+a-1)/2\leq 1$, and noting that $0<t<1$, we have $\kappa_0(\mathbb{R})\geq 2$. But $\kappa_0(\mathbb{R})\leq\kappa(\mathbb{R})\leq 2$, so $\kappa_0(\mathbb{R})=\kappa(\mathbb{R})=2$.

We will give the values of $\kappa(X)$ and $\kappa_0(X)$ for n-dimensional Euclidean space l_n^2 and infinite-dimensional Hilbert spaces H.

<u>Theorem 2.1.8</u>. *For n-dimensional Euclidean space l_n^2 ($n\geq 2$), $\kappa(l_n^2)=\kappa_0(l_n^2)=\sqrt{2}$. For infinite-dimensional Hilbert space H, $\kappa(H)=\kappa_0(H)=\sqrt{2}$.*

<u>Proof.</u> Let H be a Hilbert space (finite or infinite dimensional). We first prove $\kappa_0(H)\geq\sqrt{2}$ although this is a known result. For any $x, y\in H$ and any $0\leq t\leq 1$, noting that $\|x-y\|^2=\|x\|^2+\|y\|^2-2(x, y)$, we have

$$\begin{aligned}\|x-ty\|^2&=\|x\|^2+t^2\|y\|^2-2t(x, y)\\ &=\|x\|^2+t^2\|y\|^2+t(\|x-y\|^2-\|x\|^2-\|y\|^2)\\ &=(1-t)\|x\|^2+t\|x-y\|^2+(t^2-t)\|y\|^2\end{aligned}\tag{2.1.1}$$

For any b: $0<b<\sqrt{2}$, $b^2/2<1$, so there is t_0 such that $b^2/2<t_0<1$. Let a be such that $a^2=[2-b^2(1-t_0)]/(2t_0)$. Since $b^2<2$, $a^2>[2-2(1-t_0)]/(2t_0)=1$, so $a>1$. For any $y\in X$ with $\|y\|>1$, let $x\in\overline{B}(0, b)\cap\overline{B}(y, a)$, then from (2.1.1) we have

$$\begin{aligned}\|x-t_0y\|^2&=(1-t_0)\|x\|^2+t_0\|x-y\|^2-(t_0-t_0^2)\|y\|^2\\ &\leq(1-t_0)b^2+t_0a^2-(t_0-t_0^2)\\ &=(1-t_0)b^2+[2-b^2(1-t_0)]/2-t_0(1-t_0)\\ &=1+(b^2/2-t_0)(1-t_0)<1.\end{aligned}$$

Thus $\overline{B}(0, b)\cap\overline{B}(y, a)\subseteq\overline{B}(t_0y, 1)$. By Lemma 2.1.4, $\kappa_0(H)\geq\sqrt{2}$.

If H is an infinite-dimensional Hilbert space, then from Theorems 2.1.6

and 1.2.15, we have $\kappa(H)\le\overline{N}(H)=N(H)\le\sqrt{2}$. Thus $\kappa(H)=\kappa_0(H)=\sqrt{2}$.

For l_n^2 $(n\ge 2)$, $\kappa_0(l_n^2)\ge\sqrt{2}$ is known. We consider $\overline{B}(0,\sqrt{2})\cap\overline{B}(y,a)$, where $1<a<\sqrt{3}$ and $y=(y_1, 0, \dots, 0)$ with $y_1=\min\{a, (2-(\sqrt{(a^2-1)}-1)^2)^{1/2}\}$. Note that $\|y\|=y_1>1$ since

$$1<a^2<3 \Rightarrow 0<\sqrt{(a^2-1)}<\sqrt{2} \Rightarrow -1<\sqrt{(a^2-1)}-1<1 \Rightarrow 2-(\sqrt{(a^2-1)}-1)^2>1.$$

It is easy to verify that

$$x_{\pm}=(x_1, \pm x_2, 0, \dots, 0)\in\overline{B}(0,\sqrt{2})\cap\overline{B}(y,a),$$

where $x_1=\dfrac{2-a^2+y_1^2}{2y_1}$, $x_2=(2-x_1^2)^{1/2}$. In fact, noting that $0<x_1<1$ (since $1<y_1\le a$), we have $\|x_{\pm}\|=(x_1^2+x_2^2)^{1/2}=\sqrt{2}$, and

$$\|x_{\pm}-y\|=((x_1-y_1)^2+x_2^2)^{1/2}=(x_1^2+y_1^2-2x_1y_1+2-x_1^2)^{1/2}$$
$$=(y_1^2-2+a^2-y_1^2+2)^{1/2}=a.$$

We will show that

$$\|x_+-x_-\| = 2x_2 = 2(2-(\frac{2-a^2+y_1^2}{2y_1})^2)^{1/2}>2.$$

Since $y_1^2\le 2-(\sqrt{(a^2-1)}-1)^2$, then $0<a^2-2+y_1^2\le 2\sqrt{(a^2-1)}$. Therefore

$$2-(\frac{2-a^2+y_1^2}{2y_1})^2=\frac{8y_1^2-(2-a^2+y_1^2)^2}{4y_1^2}=\frac{4y_1^2a^2-(a^2-2+y_1^2)^2}{4y_1^2}$$
$$\ge\frac{4y_1^2a^2-4(a^2-1)}{4y_1^2}=1+\frac{4y_1^2(a^2-1)-4(a^2-1)}{4y_1^2}>1,$$

since $y_1>1$. This proves that $\|x_+-x_-\|>2$. Thus for any $z\in l_n^2$, $\overline{B}(0,\sqrt{2})\cap\overline{B}(y,a)$ can not be contained in $\overline{B}(z,1)$, since otherwise $\|x_+-x_-\|\le\|x_+-z\|+\|z-x_-\|\le 2$. By the definition of $\kappa(X)$, we obtain $\kappa(l_n^2)\le\sqrt{2}$. Thus $\kappa(l_n^2)=\kappa_0(l_n^2)=\sqrt{2}$.

Remark. The proof also gives that for any infinite-dimensional Hilbert space

H, $N(H)=\overline{N}(H)=\sqrt{2}$, which seems not to have been explicitly mentioned previously.

By Theorems 1.2.12, 1.2.14 and the Theorem above, l_n^2 $(n\geq 2)$ gives an example where $\kappa_0(X)<N(X)$ and $\kappa(X)<\overline{N}(X)$.

Next we intend to compare $\kappa(X)$ and $\kappa_0(X)$ with other numbers. We need the following definition.

Definition 2.1.9. Two Banach space X and Y are called *isomorphic* if there exists an invertible bounded linear operator from X onto Y. In case X and Y are isomorphic, *the Banach-Mazur distance from X to Y* is defined as

$$\inf\left\{\|U\|\,\|U^{-1}\| : U \text{ is bicontinuous linear operator from } X \text{ onto } Y\right\}$$

and denoted by $d(X, Y)$.

If X and Y are isomorphic Banach spaces, then (see [By-2]) the inequalities

$$N(X)\leq d(X, Y)N(Y) \text{ and } WCS(X)\leq d(X, Y)WCS(Y).$$

are true. The next example shows that neither κ nor κ_0 has this property, that is, neither of $\kappa(X)\leq d(X, Y)\kappa(Y)$ and $\kappa_0(X)\leq d(X, Y)\kappa_0(Y)$ holds in general.

Example 2.1.10. Let $E_\lambda=(l^2, |.|_\lambda)$ be l^2 space renormed by

Example 2.1.10. Let $E_\lambda=(l^2, |.|_\lambda)$ be l^2 space renormed by

where $\|.\|_2$ is the usual l^2 norm and $\|.\|_\infty$ is the usual l^∞ norm. Then for $\sqrt{5}/2\leq\lambda<\sqrt{2}$, $\kappa(E_\lambda)=\kappa_0(E_\lambda)=1$.

In fact, for any fixed $b>1$ and any a: $1<a\leq b$, since $0<\lambda^2-1<1$, there is $u_1\in\mathbb{R}$ such that

$$a/\lambda>u_1>\max\{a(\lambda^2-1)^{1/2}/\lambda,\ 1/\lambda\}. \tag{2.1.2}$$

Since $\lambda > \sqrt{5}/2$, we have

$$4\lambda^2(a^2-u_1^2) > 4\lambda^2(a^2-a^2/\lambda^2) = a^2(4\lambda^2-4) \geq a^2 > 1.$$

Hence $2\sqrt{(a^2-u_1^2)} > 1/\lambda$. Then there exists $y_2 \in \mathbb{R}$ such that

$$2\sqrt{(a^2-u_1^2)} > y_2 > 1/\lambda. \tag{2.1.3}$$

Let

$$y=(0, y_2, 0, \ldots, 0, \ldots),$$
$$u=(u_1, u_2, 0, \ldots, 0, \ldots),$$
$$w=(-u_1, u_2, 0, \ldots, 0, \ldots),$$

where $u_2=\sqrt{(a^2-u_1^2)}$. From (2.1.2), we have

$$\lambda u_2 = (\lambda^2(a^2-u_1^2))^{1/2} < (\lambda^2(a^2-a^2(\lambda^2-1)/\lambda^2))^{1/2} = a;$$

and from (2.1.3), we have $|u_2-y_2| \leq u_2$ since $0 < y_2 < 2u_2$. Hence we have

$$|u|_\lambda = |w|_\lambda = \max\left\{(u_1^2+u_2^2)^{1/2},\ \lambda|u_1|,\ \lambda|u_2|\right\} = a \leq b,$$

$$|u-y|_\lambda = |w-y|_\lambda = \max\left\{\left(u_1^2+(u_2-y_2)^2\right)^{1/2},\ \lambda|u_1|,\ \lambda|u_2-y_2|\right\} \leq a.$$

Therefore $u, w \in \overline{B}(0, b) \cap \overline{B}(y, a)$. But

$$|u-w|_\lambda = \max\{2u_1,\ \lambda(2u_1)\} = 2\lambda u_1 > 2,$$

and so $\overline{B}(0, b) \cap \overline{B}(y, a)$ cannot be contained in $\overline{B}(z, 1)$ for any $z \in E_\lambda$. Since $|y|_\lambda = \max\{y_2, \lambda y_2\} = \lambda y_2 > 1$, we obtain $\kappa(E_\lambda) = \kappa_0(E_\lambda) = 1$.

Next we prove that $d(l^2, E_\lambda) \leq \lambda$, so neither $\kappa(l^2) \leq d(l^2, E_\lambda)\kappa(E_\lambda)$ nor $\kappa_0(l^2) \leq d(l^2, E_\lambda)\kappa_0(E_\lambda)$ is true for $\sqrt{5}/2 \leq \lambda < \sqrt{2}$, since $\kappa(l^2) = \kappa_0(l^2) = \sqrt{2}$. In fact, Consider $U : l^2 \rightarrow E_\lambda$, where $Ux = x$ for any $x \in l^2$. U is a bicontinuous linear operator from l^2 onto E_λ since $\|.\|_2$ and $|.|_\lambda$ are equivalent norms:

$$\|\cdot\|_2 \leq |\cdot|_\lambda \leq \lambda\|\cdot\|_2 .$$

For any $x \in l^2$, $|Ux|_\lambda = \max\{\|x\|_2, \lambda\|x\|_\infty\} \leq \lambda\|x\|_2$, so that $\|U\| \leq \lambda$. Also for any $y \in E_\lambda$, $\|U^{-1}y\|_2 = \|y\|_2 \leq |y|_\lambda$, so $\|U^{-1}\| \leq 1$. Therefore we have $d(l^2, E_\lambda) \leq \|U\|\,\|U^{-1}\| \leq \lambda$.

Remark. Casini and Maluta [CaM] proved that $N(E_\lambda)=\sqrt{2}/\lambda$ for $1\le\lambda\le\sqrt{2}$. Thus E_λ $(\sqrt{5}/2\le\lambda<\sqrt{2})$ is also an example of a space where $N(X)>1$ but $\kappa(X)=\kappa_0(X)=1$.

In spite of the above example, we have the next result.

Proposition 2.1.11. *Let X and Y be isomorphic Banach spaces. If $d(X, Y)=1$, then $\kappa(X)=\kappa(Y)$.*

Proof. Let $U: Y\to X$ be any isomorphism. For any $x\in X$, $y\in Y$ and $r\in\mathbb{R}^+$, we use $\overline{B}_X(x, r)$ and $\overline{B}_Y(y, r)$ to denote the closed balls in X and Y respectively. It is easy to see that

$$U(\overline{B}_Y(y, r))\subseteq\overline{B}_X(Uy, r\|U\|) \quad\text{and}\quad U^{-1}(\overline{B}_X(x, r))\subseteq\overline{B}_Y(U^{-1}x, r\|U^{-1}\|).$$

For any b: $0<b<\kappa(X)$, there exists $a>1$ such that for any $u\in X$ with $\|u\|>1$, there is $z_u\in X$ such that $\overline{B}_X(0, b)\cap\overline{B}_X(u, a)\subseteq\overline{B}_X(z_u, 1)$. Since $d(X, Y)=1$, for any $\varepsilon>0$ there is an isomorphism $U: Y\to X$ such that $\|U^{-1}\|=1$ and $\|U\|<\min\{a, 1+\varepsilon\}$.

For any $w\in Y$ with $\|w\|>1$, we have $Uw\in X$ and

$$\|Uw\|=\|U^{-1}\|\|Uw\|\ge\|U^{-1}Uw\|=\|w\|>1.$$

So there is $z_{Uw}\in X$ such that $\overline{B}_X(0, b)\cap\overline{B}_X(Uw, a)\subseteq\overline{B}_X(z_{Uw}, 1)$. Thus

$$\begin{aligned}\overline{B}_Y(0, b/\|U\|)\cap\overline{B}_Y(w, a/\|U\|)&=U^{-1}U(\overline{B}_Y(0, b/\|U\|)\cap\overline{B}_Y(w, a/\|U\|))\\&\subseteq U^{-1}(\overline{B}_X(0, b)\cap\overline{B}_X(Uw, a))\\&\subseteq U^{-1}(\overline{B}_X(z_{Uw}, 1))\\&\subseteq\overline{B}_Y(U^{-1}z_{Uw}, 1),\end{aligned}$$

the last inclusion is true since $\|U^{-1}\|=1$. Since $b/\|U\|>0$ and $a/\|U\|>1$, we have $\kappa(Y)\ge b/\|U\|\ge b/(1+\varepsilon)$. By the arbitrariness of ε and b, we obtain $\kappa(Y)\ge\kappa(X)$. Since the above X and Y are arbitrary, $\kappa(X)\ge\kappa(Y)$ is also true. So $\kappa(Y)=\kappa(X)$ is proved.

Corollary 2.1.12. *For every n-dimensional Hilbert space H_n, $\kappa_0(H_n)=\kappa(H_n)=\sqrt{2}$.*

Proof. For H_n, there is an orthonormal basis $e_1, \ldots, e_n$. For every $x \in H_n$, $x=x_1e_1+\ldots+x_ne_n$, $x_i \in \mathbb{R}$, let $Tx=(x_1, \ldots, x_n)$. Then T is a bicontinuous linear operator from H_n onto l_n^2 and $\|T\|=\|T^{-1}\|=1$ since $\|Tx\|=(\sum_{i=1}^{n} x_i^2)^{1/2}=\|x\|$. Therefore, $d(H_n, l_n^2)=1$. By Proposition 2.1.11, $\kappa(H_n)=\kappa(l_n^2)$. By Theorem 2.1.8 $\kappa(l_n^2)=\sqrt{2}$, hence $\kappa_0(H_n)\leq\kappa(H_n)=\sqrt{2}$. In the proof of Theorem 2.1.8, it is shown that $\kappa_0(H_n)\geq\sqrt{2}$, so $\kappa_0(H_n)=\sqrt{2}$.

We now give some further examples.

Example 2.1.13. Let

$$l_n^\infty=\left\{x : x=(x_1, \ldots, x_n),\ x_i\in\mathbb{R}\ (1\leq i\leq n),\ |x|_\infty=\max_{1\leq i\leq n} |x_i|\right\},$$

then $\kappa(l_n^\infty)=1$ for $n\geq 2$.

In fact, for any $b>1$ and any $a>1$, let $y=(c, 0, \ldots, 0)$, where $c=\min\{a, b\}$. Note that $|y|_\infty>1$. For any $z\in l_n^\infty$, $z=(z_1, z_2, \ldots, z_n)$, let $w=(0, w_2, 0, \ldots, 0)$, where $w_2=c$ if $z_2=0$ and $w_2=-c\ \mathrm{sgn} z_2$ if $z_2\neq 0$. Then $|w|_\infty=c\leq b$ and $|w-y|_\infty=c\leq a$, that is $w\in\overline{B}(0, b)\cap\overline{B}(y, a)$. But

$$|w-z|_\infty\geq c+|z_2|>1,$$

that is $w\notin\overline{B}(z, 1)$. So $\overline{B}(0, b)\cap\overline{B}(y, a)$ can not be contained in $\overline{B}(z, 1)$ for any $z\in l_n^\infty$. Thus $\kappa(l_n^\infty)=1$.

Example 2.1.14. Let

$$l_n^1=\left\{x : x=(x_1, \ldots, x_n),\ x_i\in\mathbb{R}\ (1\leq i\leq n),\ |x|_1=\sum_{i=1}^{n} |x_i|\right\},$$

then $\kappa(l_n^1)=1$ for $n\geq 2$.

In fact, for any $b>1$ and $a>1$, let $y=(c/2, c/2, 0, \dots, 0)$, where $c=\min\{a, b\}$. Note that $|y|_1=c>1$. Let $x_1=(-c/2, c/2, 0, \dots, 0)$, and let $x_2=(c/2, -c/2, 0, \dots, 0)$. Then $|x_1|_1=|x_2|_1=c\le b$ and $|x_1-y|_1=|x_2-y|_1=c\le a$, so $x_1, x_2\in\overline{B}(0, b)\cap\overline{B}(y, a)$, but $|x_1-x_2|_1=2c>2$, thus for any $z\in l_n^1$, $\overline{B}(0, b)\cap\overline{B}(y, a)$ can not be contained in $\overline{B}(z, 1)$. Therefore $\kappa(l_n^1)=1$.

Remark. Examples 2.1.13 and 2.1.14 are examples of finite-dimensional spaces X such that $\kappa(X)=1$. Note that for any n-dimensional spaces E_n, $\overline{N}(E_n)\ge(n+1)/n$ and $\overline{N}(E_n)=2$ if and only if $E_n=l_n^{\infty}$ (Theorems 1.2.13 and 1.2.12). We see some of the difference between $\kappa(X)$ and $\overline{N}(X)$ from these two examples. Also we know that $(l_n^1)^*=l_n^{\infty}$ and $(l_n^{\infty})^*=l_n^1$, so these examples are also reflexive Banach spaces X with $\kappa(X)=1$.

2.2. Estimates for $\kappa(L^p)$

In this section, we will estimate the value of $\kappa(L^p)$ for $1<p<\infty$. We will establish an inequality in L^p $(1<p<2)$ and use it to calculate $\kappa(L^p)$. Also we will use other known inequalities to estimate the value of $\kappa(L^p)$. However we can not give the exact value of $\kappa(L^p)$.

The results in this section are included in the papers [WebZ-1] and [WebZ-2]. The establishment of the L^p $(1<p<2)$ inequality is in [WebZ-1], the estimates for $\kappa(L^p)$ are in [WebZ-2]. The paper [WebZ-1] was motivated by [XX] which claimed a similar inequality which was unfortunately false. For $1<p<2$, Xu and Xu [XX] claimed the inequality

$$(\|\lambda x+\mu y\|^2+g(\mu)\|x-y\|^2)^{1/2}\le(\lambda\|x\|^p+\mu\|y\|^p)^{1/p},\ x, y\in L^p,\ 0\le\mu\le1,\ \lambda=1-\mu,$$

where $g(\mu)=(p-1)\mu(1-\mu)$, and this $g(\mu)$ was to be the optimal function of μ

which satisfies the stated inequality. This inequality is false as is seen by taking $x=0$, $y\neq 0$ and μ close to 0. Their proof contains an error and they gave a wrong $g(\mu)$. The correct $g(\mu)$ for the inequality should be the one given in the following Lemma 2.2.2 (see Theorem 2.2.4).

In order to establish an inequality in L^p $(1<p<2)$, we need the following three Lemmas.

Lemma 2.2.1. *For any $1<p<2$, $0<\mu<1$ and $\lambda=1-\mu$, let*

$$f(x)=(\mu+\lambda x^{p-1})(\mu+\lambda x^p)^{2/p-1}-(\mu+\lambda x),\ 0\leq x\leq 1\ .$$

Then, if $1/2\leq\mu<1$, $f(x)\leq 0$ for all $x\in[0,1]$. If $0<\mu<1/2$, $f(x)=0$ has a unique solution in $(0,1)$; in fact the solution is $x=(\mu/\lambda)^{2/p}$, which we denote by a or $a(\mu)$.

Proof. It is routine to obtain

$$f'(x)=(\mu+\lambda x^p)^{2/p-2}[\lambda^2x^{2p-2}+(2-p)\lambda\mu x^{p-1}+(p-1)\lambda\mu x^{p-2}]-\lambda$$

and

$$f''(x)=(p-1)(2-p)\lambda\mu(1-x)x^{p-3}(\mu+\lambda x^p)^{2/p-3}(\lambda x^p-\mu)\ .$$

If $1/2\leq\mu<1$, it is readily seen that $f''(x)<0$ for all $x\in(0,1)$ since $\lambda x^p-\mu<\lambda-\mu\leq 0$, that is f' is decreasing on $(0,1)$, so $f'(x)>f'(1)=0$ for all $x\in(0,1)$, which implies that f is increasing on $(0,1)$, hence $f(x)<f(1)=0$ for all $x\in(0,1)$. Therefore $f(x)\leq 0$ for all $x\in[0,1]$.

If $0<\mu<1/2$, then $f''(x)<0$ when $0<x<(\mu/\lambda)^{1/p}$, $f''(x)=0$ when $x=(\mu/\lambda)^{1/p}$, and $f''(x)>0$ when $(\mu/\lambda)^{1/p}<x<1$, that is f' is decreasing on $(0,(\mu/\lambda)^{1/p})$ and increasing on $((\mu/\lambda)^{1/p},1)$, also $\min\limits_{0<x<1} f'(x)=f'\left((\mu/\lambda)^{1/p}\right)$. Hence $f'(x)<f'(1)=0$ if $(\mu/\lambda)^{1/p}\leq x<1$, and $\lim\limits_{x\to 0}f'(x)=+\infty$. Therefore there exists a unique $b\in(0,(\mu/\lambda)^{1/p})$ such that $f'(b)=0$ and $f'(x)>0$ if $0<x<b$, $f'(x)<0$ if

$b<x<1$. Since f is increasing in $(0, b)$ and decreasing in $(b, 1)$, and $\max_{0<x<1} f(x)=f(b)$, we have $f(x)>f(1)=0$ if $b\leq x<1$. Also $f(0)=\mu^{2/p}-\mu<0$, hence there is a unique $a\in(0, b)$ such that $f(a)=0$. Since

$$\begin{aligned}
f((\mu/\lambda)^{2/p}) &= (\mu+\lambda(\mu/\lambda)^{2(p-1)/p})(\mu+\lambda(\mu/\lambda)^2)^{2/p-1}-(\mu+\lambda(\mu/\lambda)^{2/p}) \\
&= (\mu+\lambda(\mu/\lambda)^{2-2/p})(\mu/\lambda)^{2/p-1}-(\mu+\lambda(\mu/\lambda)^{2/p}) \\
&= (\lambda(\mu/\lambda)^{2/p}+\lambda(\mu/\lambda))-(\mu+\lambda(\mu/\lambda)^{2/p}) = 0\,,
\end{aligned}$$

we have $a=(\mu/\lambda)^{2/p}$ since the solution of $f(x)=0$ is unique.

Lemma 2.2.2. *For $1<p<2$, $0\leq\mu\leq 1$, $\lambda=1-\mu$, and all real numbers x, y, we have the following inequality*

$$(|\lambda x+\mu y|^2+g(\mu)|x-y|^2)^{1/2}\leq(\lambda|x|^p+\mu|y|^p)^{1/p}, \tag{2.2.1}$$

where g is given by

$$g(\mu)=\frac{\mu^{2/p}\lambda^2-\lambda^{2/p}\mu^2}{\lambda^{2/p}-\mu^{2/p}}$$

for $\mu\neq 1/2$, and $g(1/2)=(p-1)/4$. Moreover, g is the best possible non-negative function in the sense that $g(\mu)\geq f(\mu)$ for all $\mu: 0\leq\mu\leq 1$, for any other non-negative function $f(\mu)$ which satisfies (2.2.1).

Proof. If $x=y=0$ or $x=y=1$, (2.2.1) is true for any $g(\mu)$. If $|y|\geq|x|$ and $y\neq 0$, then (2.2.1) is equivalent to

$$(|\lambda(x/y)+\mu|^2+g(\mu)|(x/y)-1|^2)^{1/2}\leq(\lambda|x/y|^p+\mu)^{1/p}.$$

If $|y|<|x|$, then (2.2.1) is equivalent to

$$(|\lambda+\mu(y/x)|^2+g(\mu)|1-(y/x)|^2)^{1/2}\leq(\lambda+\mu|y/x|^p)^{1/p}.$$

Since x and y are arbitrary and λ and μ can be exchanged in (2.2.1), without loss of generality, we can suppose that $y=1$ and $-1\leq x<1$. Then it is easy to see that the greatest possible function for (2.2.1) to be valid is

$$G(\mu)=\inf_{-1\le x<1}\min\left\{\frac{(\mu+\lambda|x|^p)^{2/p}-(\mu+\lambda x)^2}{(1-x)^2},\ \frac{(\lambda+\mu|x|^p)^{2/p}-(\lambda+\mu x)^2}{(1-x)^2}\right\},$$

Obviously, $G(\mu)=G(\lambda)$ and G is symmetric about $\mu=1/2$, we may consider only the case $0\le\mu\le1/2$. We prove that $G(\mu)=g(\mu)$ for $0\le\mu\le1/2$. To show this, for every $-1\le x<1$, we consider

$$F(\mu)=(\lambda+\mu|x|^p)^{2/p}-(\lambda+\mu x)^2-(\mu+\lambda|x|^p)^{2/p}+(\mu+\lambda x)^2,\ 0\le\mu\le1/2.$$

It is easy to see that $F(0)=F(1/2)=0$. For $0<\mu<1/2$, we have

$$\begin{aligned}F'(\mu)&=\frac{2}{p}(-1+|x|^p)(\lambda+\mu|x|^p)^{2/p-1}-2(x-1)(\lambda+\mu x)\\&\quad-\frac{2}{p}(1-|x|^p)(\mu+\lambda|x|^p)^{2/p-1}+2(1-x)(\mu+\lambda x),\end{aligned}$$

and

$$F''(\mu)=\frac{2}{p}\left(\frac{2}{p}-1\right)(1-|x|^p)^2\left((\lambda+\mu|x|^p)^{2/p-2}-(\mu+\lambda|x|^p)^{2/p-2}\right).$$

Since $1<p<2$ and $\lambda+\mu|x|^p\ge\mu+\lambda|x|^p$ for $0\le\mu\le1/2$ and $|x|\le1$, it follows that $F''(\mu)\le0$. Thus $F(\mu)\ge0$ for all $0\le\mu\le1/2$. Therefore

$$G(\mu)=\inf_{-1\le x<1}\frac{(\mu+\lambda|x|^p)^{2/p}-(\mu+\lambda x)^2}{(1-x)^2}.$$

From this expression we see that $G(0)=0$. For $0<\mu\le1/2$, let

$$H(x)=\frac{(\mu+\lambda|x|^p)^{2/p}-(\mu+\lambda x)^2}{(1-x)^2},\ -1\le x<1.$$

Then for $0<x<1$ and $-1\le x<0$, we obtain

$$H'(x)=2(1-x)^{-3}[(\mu+\lambda|x|^{p-1}\mathrm{sgn}(x))(\mu+\lambda|x|^p)^{2/p-1}-(\mu+\lambda x)].$$

If $-1\le x<0$, $H'(x)\le0$, so $H(x)\ge H(0)$ for any $-1\le x<0$, hence $G(\mu)=\inf_{0\le x<1}H(x)$. When $\mu=1/2$, $H'(x)=2(1-x)^{-3}[(\frac{1}{2}+\frac{1}{2}x^{p-1})(\frac{1}{2}+\frac{1}{2}x^p)^{2/p-1}-(\frac{1}{2}+\frac{1}{2}x)]\le0$ for all $0<x<1$, hence $H(x)\ge H(1-)$ for $0\le x<1$, i.e. $G(1/2)=H(1-)$, and $H(1-)=\lim_{x\to1-}H(x)=\frac{1}{4}(p-1)$, therefore $G(1/2)=\frac{1}{4}(p-1)=g(1/2)$. When $0<\mu<1/2$, for $0\le x<1$, $H'(x)=2(1-x)^{-3}f(x)$, where $f(x)$ is the function given in Lemma 2.2.1. From Lemma 2.2.1 we have, for

$a=(\mu/\lambda)^{2/p}$, $H'(x)<0$ if $0<x<a$, $H'(a)=0$, and $H'(x)>0$ if $a<x<1$, that is $\min\limits_{0\le x<1} H(x)=H(a)$. Therefore

$$G(\mu) = H(a) = \frac{(\mu+\lambda a^p)^{2/p}-(\mu+\lambda a)^2}{(1-a)^2} = \frac{a-(\mu+\lambda a)^2}{(1-a)^2}$$

$$= \frac{a\lambda^2-\mu^2}{1-a} = \frac{\mu^{2/p}\lambda^2-\mu^2\lambda^{2/p}}{\lambda^{2/p}-\mu^{2/p}} = g(\mu).$$

Note that this gives some alternative expressions for g.

The next Lemma may be found in Beauzamy [Bea] (Lemma 11, p. 201–202).

Lemma 2.2.3. *If $1<p<2$, for every finite sequence (x_j) of elements of L^p, we have*

$$(\sum_j \|x_j\|^2)^{1/2} \le \|(\sum_j |x_j|^2)^{1/2}\| .$$

Now we can obtain an L^p inequality for $1<p<2$.

Theorem 2.2.4. *For $1<p<2$ the inequality*

$$(\|\lambda x+\mu y\|^2+g(\mu)\|x-y\|^2)^{1/2} \le (\lambda\|x\|^p+\mu\|y\|^p)^{1/p} \tag{2.2.2}$$

holds for all $x, y\in L^p$ and $0\le\mu\le 1$, $\lambda=1-\mu$, where g is the function given in Lemma 2.2.2.

Proof. From Lemmas 2.2.3 and 2.2.2, we have

$$\begin{aligned}(\|\lambda x+\mu y\|^2+g(\mu)\|x-y\|^2)^{1/2} &\le \|(|\lambda x+\mu y|^2+g(\mu)|x-y|^2)^{1/2}\| \\ &\le \|(\lambda|x|^p+\mu|y|^p)^{1/p}\| \\ &= (\lambda\|x\|^p+\mu\|y\|^p)^{1/p}.\end{aligned}$$

Next we use the inequality established in Theorem 2.2.4 to estimate $\kappa(L^p)$ $(1<p<2)$.

Theorem 2.2.5. *For* $1<p<2$, *we have* $\kappa(L^p)\geq\kappa_0(L^p)\geq M^{1/p}$, *where*

$$M=\max_{1/2\leq t\leq 1}\frac{(1+g(t))^{p/2}-t}{1-t}$$

and g is the function given in Lemma 2.2.2.

Proof. By Theorem 2.2.4, for any $t\in[0,1]$ and $x\in\overline{B}(0,b)\cap\overline{B}(y,a)$ where $\|y\|>1$, we have

$$\begin{aligned}\|x-ty\|^2&=\|(1-t)x+t(x-y)\|^2\\ &\leq(t\|x-y\|^p+(1-t)\|x\|^p)^{2/p}-g(t)\|y\|^2 &(2.2.3)\\ &\leq[ta^p+(1-t)b^p]^{2/p}-g(t).\end{aligned}$$

We want to choose suitable b, a and t such that the right hand side can be less than or equal to 1. Let

$$M(t)=\frac{(1+g(t))^{p/2}-t}{1-t}$$

and let M be its maximum on $[1/2,1]$. Since $M(t)=1+\frac{(1+g(t))^{p/2}-1}{1-t}$ and $g(t)=g(1-t)$, $M(t)\leq M(1-t)$ for $0\leq t\leq 1/2$. Therefore $M=\max\limits_{0\leq t\leq 1}M(t)$. For any $b<M^{1/p}$, there is t_0: $1/2\leq t_0<1$ such that $b^p<M(t_0)$. Then $t_0+(1-t_0)b^p<(1+g(t_0))^{p/2}$. Therefore there is $a>1$ so that $t_0a^p+(1-t_0)b^p<(1+g(t_0))^{p/2}$, that is $(t_0a^p+(1-t_0)b^p)^{2/p}-g(t_0)<1$. Thus for all $y\in L^p$ with $\|y\|>1$, we have $\overline{B}(0,b)\cap\overline{B}(y,a)\subseteq\overline{B}(t_0y,1)$. By Theorem 2.1.4, we have $\kappa_0(L^p)\geq M^{1/p}$.

Although we have an explicit formula for g, and it is simple to find M numerically for any p, we do not have a formula for the maximum M.

Smarzewski [Sm] recently proved an inequality in an abstract L^p $(1<p<2)$ space X_p which enables us to give an explicit lower bound for $\kappa(X_p)$. Recall that for $1\leq p<\infty$, a Banach lattice X_p for which $\|x+y\|^p=\|x\|^p+\|y\|^p$ whenever x, $y\in X$

and $x \wedge y=0$, is called an abstract L^p space [LinT-1], p.14. Here $x \wedge y$ denotes the greatest lower bound of x and y. It is obvious that every L^p space is an abstract L^p space.

Lemma 2.2.6. [Sm] *Let X_p be an abstract L^p space with $1<p<2$, then*

$$\|(1-t)x+ty\|^2 \leq (1-t)\|x\|^2+t\|y\|^2-(p-1)t(1-t)\|x-y\|^2$$

for all $x, y \in X_p$ and $0<t<1$.

Theorem 2.2.7. *If X_p is an abstract L^p space with $1<p<2$, then $\kappa(X_p) \geq \kappa_0(X_p) \geq \sqrt{p}$.*

Proof. By Lemma 2.2.6, for any $t \in [0, 1]$ and $x \in \overline{B}(0, b) \cap \overline{B}(y, a)$ where $\|y\|>1$, we have

$$\begin{aligned} \|x-ty\|^2 &= \|(1-t)x+t(x-y)\|^2 \\ &\leq (1-t)\|x\|^2+t\|x-y\|^2-(p-1)t(1-t)\|y\|^2 \\ &\leq (1-t)b^2+ta^2-(p-1)t(1-t). \end{aligned}$$

For any $b<\sqrt{p}$, there is a $t_0: 0<t_0<1$ so that $b^2<(p-1)t_0+1$, that is, $(1-t_0)b^2+t_0-(p-1)t_0(1-t_0)<1$. Thus there is $a>1$ so that

$$(1-t_0)b^2+t_0a^2-(p-1)t_0(1-t_0)<1.$$

Therefore, $\overline{B}(0, b) \cap \overline{B}(y, a) \subseteq \overline{B}(t_0 y, 1)$. By Lemma 2.1.4, the conclusion of the Theorem follows.

Remark. Numerical calculations show the lower bound $\sqrt{p}$ to be better then the one obtained in Theorem 2.2.5 .

To obtain a lower bound for $\kappa(L^p)$ $(p>2)$, we need the following two lemmas established in [Lim-1].

Lemma 2.2.8. (Lim) *Let $p>2$. The following inequality*

$$\|\lambda x+\mu y\|^p+g(\mu)\|x-y\|^p\leq\lambda\|x\|^p+\mu\|y\|^p,$$

is valid for $x, y\in L^p$ and $0\leq\mu\leq 1$ with $\lambda=1-\mu$, where $g(\mu)=\lambda\mu h(\mu)$, $h(\mu)=\dfrac{1+[x(\mu\wedge\lambda)]^{p-1}}{[1+x(\mu\wedge\lambda)]^{p-1}}$ where $\mu\wedge\lambda=\min\{\mu,\lambda\}$ and $x(\mu)$ is the unique solution of the equation $\lambda x^{p-1}-\mu-(\lambda x-\mu)^{p-1}=0$ $(0<\mu\leq 1/2)$. Moreover $g(\mu)$ is the best possible in the sense that it is larger then or equal to any other such function, in particular, $g(\mu)\geq\lambda^{p/2}\mu^{p/2}$.

Lemma 2.2.9. (Lim) *For the function $h(\mu)$ given in Lemma 2.2.8,*

$$\sup_{0<\mu\leq 1}(\mu h(\mu))=\frac{1+\alpha^{p-1}}{(1+\alpha)^{p-1}},$$

where α is the unique number in $[0, 1]$ that satisfies $(p-2)\alpha^{p-1}+(p-1)\alpha^{p-2}-1=0$.

Now we give a lower bound for $\kappa(L^p)$ $(p>2)$.

Theorem 2.2.10. *For $p>2$, we have*

$$\kappa(L^p)\geq\kappa_0(L^p)\geq\left(1+\frac{1+\alpha^{p-1}}{(1+\alpha)^{p-1}}\right)^{1/p},$$

where α is the number given in Lemma 2.2.9.

Proof. By Lemma 2.2.8, for any $t\in[0, 1]$ and $x\in\overline{B}(0, b)\cap\overline{B}(y, a)$ where $\|y\|>1$, we have

$$\begin{aligned}\|x-ty\|^p&=\|(1-t)x+t(x-y)\|^p\\ &\leq t\|x-y\|^p+(1-t)\|x\|^p-g(t)\|y\|^p\\ &\leq ta^p+(1-t)b^p-g(t).\end{aligned}\tag{2.2.3}$$

Now let $f(t)=\dfrac{1+g(t)-t}{1-t}=1+th(t)$, then by Lemma 2.2.9, we have

$$S := \sup_{0 \le t \le 1} f(t) = 1 + \sup_{0 < t \le 1} (th(t)) = 1 + \frac{1+\alpha^{p-1}}{(1+\alpha)^{p-1}}.$$

For any $b < S^{1/p}$, there is $t_0 : 0 < t_0 < 1$ such that $b^p < f(t_0)$. Then

$$t_0 + (1-t_0)b^p - g(t_0) < 1.$$

Therefore there is $a > 1$ so that $t_0 a^p + (1-t_0)b^p - g(t_0) < 1$. Thus for all $y \in L^p$ with $\|y\| > 1$, we have $\overline{B}(0, b) \cap \overline{B}(y, a) \subseteq \overline{B}(t_0 y, 1)$ for above b, a and t_0. This proves $\kappa_0(L^p) \ge S^{1/p}$ by Theorem 2.1.4.

2.3. The fixed point theorems of uniformly Lipschitzian mappings

First we give a slight generalization of the fixed point theorem of Lifschitz (Theorem 1.2.19). We just replace $n \ge 1$ with $n \ge n_0$ for some $n_0 \in \mathbb{N}$ in condition 1) of Theorem 1.2.19. Also the proof is similar as that of Lifschitz (see [KrZ], p.227).

Theorem 2.3.1. *Let (M, ρ) be a complete metric space and let f be a mapping from M into itself. Assume that*

1) *There are $k < \kappa(M)$ and $n_0 \in \mathbb{N}$ so that for any $x, y \in M$ and $n \ge n_0$, we have $\rho(f^n x, f^n y) \le k\rho(x, y)$;*

2) *There is $x_0 \in M$ so that $\{f^n x_0\}$ is bounded.*

Then f has at least one fixed point in M.

Proof. For any $y \in M$, let

$$R(y) = \inf\left\{d \ge 0 : \text{there is } x \in M \text{ such that } \{f^n x\}_{n \ge n_0} \subseteq \overline{B}(y, d)\right\}.$$

Then from condition 2), $R(y) < +\infty$ for any $y \in M$. It is easy to check that $R(y) = 0$ if and only if $fy = y$. In fact, if $fy = y$, then for any $\varepsilon > 0$, $\{f^n y\}_{n \ge n_0} \subseteq \overline{B}(y, \varepsilon)$, so $R(y)=0$. Conversely if $R(y)=0$, then for any $\varepsilon>0$, there is $x\in M$ so that $\{f^n x\}_{n\geq n_0}\subseteq\overline{B}(y,\varepsilon)$. Hence, for any $n\geq n_0$, from condition 1), we have

$$\rho(f^n y, y)\leq\rho(f^n y, f^{2n}x)+\rho(f^{2n}x, y)$$
$$\leq k\rho(y, f^n x)+\varepsilon\leq(k+1)\varepsilon.$$

Then $f^n y=y$. Therefore $fy=f^{n+1}y=y$.

Next we prove there exists $y\in M$ so that $R(y)=0$. For $b: k<b<\kappa(M)$, there is $a>1$ such that for any $x, y\in M$ and $r>0$, $\rho(x,y)>r$ implies there exists $z\in M$ so that $\overline{B}(x, br)\cap\overline{B}(y, ar)\subseteq\overline{B}(z, r)$. Let $\lambda: 0<\lambda<1$ be such that $\gamma=\min\{a\lambda, b\lambda/k\}>1$ (since $a>1$ and $b/k>1$, such λ exists). We construct a sequence $\{y_p\}\subseteq M$ such that for any $p\in\mathbb{N}$,

$$R(y_{p+1})\leq\lambda R(y_p) \quad\text{and}\quad \rho(y_{p+1}, y_p)\leq(\lambda+\gamma)R(y_p).$$

Let y_1 be any point in M. When $y_1,\ldots,y_p$ are given, if $R(y_p)=0$, let $y_{p+1}=y_p$; if $R(y_p)>0$, there is $j\geq n_0$ such that $\rho(f^j y_p, y_p)>\lambda R(y_p)$. From the definition of $R(y_p)$, there is $x\in M$ such that $\{f^n x\}_{n\geq n_0}\subseteq\overline{B}(y_p, \gamma R(y_p))$. Let $\hat{x}=f^j x$, then for any $n\geq n_0$,

$$f^n\hat{x}=f^{n+j}x\in\overline{B}(y_p, \gamma R(y_p))\subseteq\overline{B}(y_p, a\lambda R(y_p))$$

and from condition 1), we have

$$\rho(f^n\hat{x}, f^j y_p)=\rho(f^{j+n}x, f^j y_p)\leq k\rho(f^n x, y_p)\leq k\gamma R(y_p)\leq b\lambda R(y_p).$$

Therefore, $\{f^n\hat{x}\}_{n\geq n_0}\subseteq\overline{B}(y_p, a\lambda R(y_p))\cap\overline{B}(f^j y_p, b\lambda R(y_p)):=D$. There is $w\in M$ so that $D\subseteq\overline{B}(w, \lambda R(y_p))$, hence $\{f^n\hat{x}\}_{n\geq n_0}\subseteq\overline{B}(w, \lambda R(y_p))$. Therefore, $R(w)\leq\lambda R(y_p)$. Let $y_{p+1}=w$, then $R(y_{p+1})\leq\lambda R(y_p)$ and

$$\rho(y_{p+1}, y_p)\leq\rho(y_{p+1}, f^n\hat{x})+\rho(f^n\hat{x}, y_p)$$
$$\leq\lambda R(y_p)+\gamma R(y_p)=(\lambda+\gamma)R(y_p).$$

Since $\lambda<1$, $R(y_p)\leq\lambda^{p-1}R(y_1)\longrightarrow 0$ $(p\longrightarrow\infty)$. Also for any $p, l\in\mathbb{N}$,

$$\rho(y_p, y_{p+l})\leq\sum_{i=0}^{l-1}\rho(y_{p+i}, y_{p+i+1})\leq\sum_{i=0}^{l-1}(\lambda+\gamma)R(y_{p+i})$$

$$\leq(\lambda+\gamma)\sum_{i=0}^{l-1}\lambda^{p+i-1}R(y_1)\leq(\lambda+\gamma)\lambda^{p-1}/(1-\lambda)R(y_1),$$

so $\{y_p\}$ is a Cauchy sequence and converges to a point $y\in M$. We will see that $R(y)=0$. In fact, for any $\varepsilon>0$, there is a y_p so that $R(y_p)<\varepsilon/2$ and $\rho(y_p, y)<\varepsilon/2$. There exists $x\in M$ such that $\{f^n x\}_{n\geq n_0}\subseteq\overline{B}(y_p, \varepsilon/2)$, then we have $\{f^n x\}_{n\geq n_0}\subseteq\overline{B}(y, \varepsilon)$. Hence $R(y)=0$.

Remark. In [GorK], Gornicki and Kruppel claimed the following fixed point theorem: Let (M,ρ) be a complete metric space and $A\subseteq\mathbb{N}$ be a subset with a Banach density $\mu(A)>1/2$, where $\mu(A)=\mathrm{LIM}\,(|A\cap N_n|/n)$, LIM denotes the Banach limit, $N_n=\{1, 2, \ldots, n\}$, $|A\cap N_n|$ denotes the number of elements of $A\cap N_n$. Let $f: M\longrightarrow M$ satisfy:

1) There is $k<\kappa(M)$ so that $\rho(f^i x, f^i y)\leq k\rho(x, y)$ for any $i\in A$ and $x, y\in M$;

2) There is $x_0\in M$ such that $\{f^i x_0\}_{i\in A}$ is bounded.

Then f has a fixed point in M.

However, their proof seems to be wrong and it seems the above result cannot be obtained by the method given in their proof. We do not know whether this result is true or not. Here we just point out the errors in the proof.

First, they said: For $b\in(k, \kappa(M))$, there is $a>1$ such that for any $u, v\in M$ and $r>0$, $\rho(u, v)>r$ implies that there is $w\in M$ so that

$$\overline{B}(u, br)\cap\overline{B}(v, ar)\subseteq\overline{B}(w, r).$$

Let $\lambda\in(0, 1)$ be such that $\gamma=\min\{\lambda ak^{-1}, \lambda bk^{-1}\}>1$ ...... We can see that if $k>1$, $a\leq k$ can be true, so in this case, $\gamma\leq\lambda ak^{-1}\leq\lambda<1$ for any $\lambda\in(0, 1)$.

Secondly, they said: There exists $n\in A$ such that $\rho(f^n y_m, y_m)>\lambda r(y_m)$ (where $r(x)$ is like our $R(x)$). As $\gamma>1$ there exists $x\in M$ such that $\{f^i x\}_{i\in A}\subseteq\overline{B}(y_m, \gamma r(y_m))$. Put $\hat{x}=f^n x$. Then $\{f^i\hat{x}\}_{i\in A}\subseteq\overline{B}(y_m, k\gamma r(y_m))$. But why is the last inclusion true? I think they may need $n+A\subseteq A$, then $\{f^i\hat{x}\}_{i\in A}\subseteq\{f^i x\}_{i\in A}\subseteq\overline{B}(y_m, \gamma r(y_m))$. If so, the condition for any $n\in A$, $n+A\subseteq A$ should be added. Also $\mu(A)>1/2$ implies $(1+A)\cap A\neq\emptyset$. So if $m\in(1+A)\cap A$, it is easy to see that

$$A\supseteq\left\{(p+k)m+k:\ k\in\{1, 2, \ldots, m\},\ p=1, 2, 3, \ldots\right\}.$$

But for $n>m^2+m$, $n=lm+k$ with $l\in\mathbb{N}$ and $k\in\{1, 2, \ldots, m\}$. Since $l>m\geq k$, $l=p+k$ with $p\in\mathbb{N}$, we have $n=(p+k)m+k\in A$. Therefore $\mu(A)=1$.

Next we give another fixed point Theorem on uniformly k-Lipschitzian mapping.

Theorem 2.3.2. *Let X be a Banach space and let C be a nonempty, closed, bounded and convex subset of X. Let f be a mapping from C into itself. If there are $m\in\mathbb{N}$ and $k\in\mathbb{R}^+$ with $k<\kappa_0(X)$ so that*

$$\|f^{mn}x-f^{mn}y\|\leq k\|x-y\| \text{ for all } n\in\mathbb{N} \text{ and } x, y\in C,$$

then f^m has at least one fixed point in C. Let $F(f^m)$ denote the fixed point set of f^m. If X is strictly convex and f satisfies

1) $\|f^m x-p\|\leq\|x-p\|$ *for any $p\in F(f^m)$ and $x\in C$, and*

2) *there is $k_1\in\mathbb{R}^+$ with $k_1<\kappa_0(X)$ such that*

$$\|f^j x-f^j y\|\leq k_1\|x-y\| \text{ for all } x, y\in F(f^m) \text{ and } j=1, 2, \ldots, m-1,$$

then f has at least one fixed point in C.

Proof. Since $k<\kappa_0(X)\leq\kappa(C)$, $f^m: C\rightarrow C$ satisfies the conditions of Theorem 2.3.1, then f^m has at least one fixed point. If f^m has only one fixed point

x_0, then $fx_0=x_0$. Now suppose $\mathrm{diam}(F(f^m))>0$.

If X is strictly convex and f satisfies condition 1), then $f: F(f^m)\to F(f^m)$ and $F(f^m)$ is closed and convex. In fact, for any $x\in F(f^m)$, we have $f^m(fx)=f(f^m x)=fx$, so $fx\in F(f^m)$. For any $x_n\in F(f^m)$ with $x_n\to x_0\in C$ $(n\to\infty)$, we have

$$\begin{aligned}\|f^m x_0-x_0\| &= \|(f^m x_0-f^m x_n)+(x_n-x_0)\| \\ &\le \|f^m x_0-f^m x_n\|+\|x_n-x_0\|\le (k+1)\|x_n-x_0\|,\end{aligned}$$

so $f^m x_0=x_0$. Hence $F(f^m)$ is closed. Next we prove $F(f^m)$ is convex. Let x and y be two point of $F(f^m)$ and let $z=(x+y)/2$. Then $\|f^m z-x\|\le\|z-x\|=\|x-y\|/2$ and $\|y-f^m z\|\le\|z-y\|=\|x-y\|/2$. If $\|(f^m z-x)-(y-f^m z)\|>0$, then

$$\|(f^m z-x)+(y-f^m z)\|/2<\|x-y\|/2$$

as X is strictly convex, this is a contradiction. Hence $\|(f^m z-x)-(y-f^m z)\|=0$, that is, $f^m z=(x+y)/2=z$. Therefore $F(f^m)$ is convex.

For any $x, y\in F(f^m)$ and any $n\in\mathbb{N}: n=pm+j, p\in\{0\}\cup\mathbb{N}, j\in\{1, 2, \ldots, m-1\}$, note that $f^{pm}x=x$ and $f^{pm}y=y$, we have

$$\begin{aligned}\|f^n x-f^n y\| &= \|f^{pm+j}x-f^{pm+j}y\| \\ &= \|f^j(f^{pm}x)-f^j(f^{pm}y)\|=\|f^j x-f^j y\|\le k_1\|x-y\|.\end{aligned}$$

Also $k_1<\kappa_0(X)\le\kappa(F(f^m))$, so f has at least one fixed point in $F(f^m)$.

Remark. Since we know $\kappa_0(H)=\kappa(H)=\sqrt{2}$ for Hilbert space H and we know lower bound for $\kappa_0(L^p)$, we can obtain fixed point theorems for Hilbert and L^p spaces as consequence of Theorems 2.3.1 and 2.3.2.

Some authors such as Lim [Lim-1] and Smarzewski [Sm] proved the fixed point theorems on uniformly k-Lipschitzian mapping in L^p space by using a direct proof. In fact, their results are consequences of the fixed point theorem of Lifschitz by using the lower bounds for $\kappa_0(L^p)$ in section 2. From this we can see the importance of giving the exact values of $\kappa_0(L^p)$ or $\kappa(L^p)$. However this seems not to be easy.

Now we give a theorem on weak convergence for the fixed point.

<u>Theorem 2.3.3.</u> *Let X be a separable, uniformly convex Banach space with X^* uniformly convex. Let $f : X \longrightarrow X$ satisfy that*

1) *For any $x, y \in X$, we have $\|fx - fy\| \le \|x-y\|$;*

2) *There is $x_0 \in X$ so that $\{f^n x_0\}$ is bounded.*

Then, the fixed point set $F(f)$ of f is not empty. For any $x_0 \in X$, let

$$x_{n+1} = (1-c_n)x_n + c_n f x_n, \quad 0 < a \le c_n \le b < 1.$$

then there is $p \in F(f)$ such that $w\text{-}\lim_{n\to\infty} F(x_n - p) = 0$. In fact, $\phi(p) = \min_{z \in X} \phi(z)$, where $\phi(z) = \limsup_{n\to\infty} \|x_n - z\|$.

<u>Proof.</u> Since X is uniformly convex, $\kappa(X) > 1$. Also f is a uniformly 1-Lipschitzian mapping. So by Theorem 1.2.19, f has at least one fixed point. For any $p \in F(f)$, from condition 1), we have

$$\begin{aligned}\|x_{n+1}-p\| &= \|(1-c_n)(x_n-p)+c_n(fx_n-p)\| \\ &\le (1-c_n)\|x_n-p\| + c_n\|fx_n - fp\| \le \|x_n-p\|.\end{aligned}$$

So $\lim_{n\to\infty} \|x_n - p\| := d_x$ exists. If $d_x = 0$, there is nothing left to prove. Now suppose $d_x \ne 0$, and let $w_n = (x_n-p)/\|x_n-p\|$, $y_n = (fx_n-p)/\|x_n-p\|$. Then $\|w_n\| \le 1$, $\|y_n\| \le 1$ and

$$\|(1-c_n)w_n + c_n y_n\| = \|x_{n+1}-p\| / \|x_n - p\| \longrightarrow 1 \ (n \longrightarrow \infty).$$

By Lemma 1.2.2, $\|w_n - y_n\| \longrightarrow 0$ $(n \longrightarrow \infty)$, that is, $\|x_n - fx_n\| / \|x_n - p\| \longrightarrow 0$

$(n \longrightarrow \infty)$, so we obtain $\|x_n - f x_n\| \longrightarrow 0$ $(n \longrightarrow \infty)$.

For any subsequence $\{x_k\}$ (say) of $\{x_n\}$, by Lemma 1.2.3, there is a subsequence $\{x_j\}$ (say) of $\{x_k\}$ such that $\phi(z) = \lim_{j\to\infty} \|x_j - z\|$ exists for all $z \in X$, and there is a unique $v \in X$ so that $\phi(v) = \min_{z\in X} \phi(z)$ and $w\text{-}\lim_{j\to\infty} F(x_j - v) = 0$. For any j, since

$$\|x_{j+1} - z\| = \|(1-c_j)x_j + c_j f x_j - z\| = \|(x_j - z) - c_j(x_j - f x_j)\|,$$

we have

$$\|x_j - z\| - c_j\|x_j - f x_j\| \le \|x_{j+1} - z\| \le \|x_j - z\| + c_j\|x_j - f x_j\|.$$

So $\lim_{j\to\infty} \|x_{j+1} - z\| = \lim_{j\to\infty} \|x_j - z\| = \phi(z)$ for any $z \in X$, as $\|x_j - f x_j\| \longrightarrow 0$ $(j \longrightarrow \infty)$. Since

$$\begin{aligned}
\|x_{j+1} - f v\| &= \|(1-c_j)(x_j - f v) + c_j(f x_j - f v)\| \\
&\le (1-c_j)\|x_j - f v\| + c_j\|f x_j - f v\| \\
&\le (1-c_j)\|x_j - f v\| + c_j\|x_j - v\|,
\end{aligned}$$

we obtain

$$(\|x_{j+1} - f v\| - \|x_j - f v\|)/c_j + \|x_j - f v\| \le \|x_j - v\|.$$

Let $j \longrightarrow \infty$, we have, $\phi(f v) \le \phi(v)$. Hence $\phi(f v) = \phi(v)$. Since the infimum of ϕ is unique, $f v = v$.

If v is independent of the subsequence, the proof is finished.

Suppose $\{x_i\}$ and $\{x_m\}$ (say) are two subsequences of $\{x_n\}$ such that $\phi_1(z) = \lim_{i\to\infty} \|x_i - z\|$ exists for all $z \in X$, and there exists unique $v_1 \in X$ so that $\phi_1(v_1) = \min_{z\in X} \phi_1(z)$; also $\phi_2(z) = \lim_{m\to\infty} \|x_m - z\|$ exists for all $z \in X$, and there exists unique $v_2 \in X$ so that $\phi_2(v_2) = \min_{z\in X} \phi_2(z)$. We know that both v_1 and v_2 are fixed points of f, so both $\lim_{n\to\infty} \|x_n - v_1\|$ and $\lim_{n\to\infty} \|x_n - v_2\|$ exist. Hence we have

$$
\begin{aligned}
\phi_2(v_1) &= \lim_{m\to\infty}\|x_m - v_1\| = \lim_{n\to\infty}\|x_n - v_1\| \\
&= \lim_{i\to\infty}\|x_i - v_1\| = \phi_1(v_1) \le \phi_1(v_2) \\
&= \lim_{i\to\infty}\|x_i - v_2\| = \lim_{n\to\infty}\|x_n - v_2\| \\
&= \lim_{m\to\infty}\|x_m - v_2\| = \phi_2(v_2)\,.
\end{aligned}
$$

Hence $v_1 = v_2$.

Remark. Let X be a Banach space. If $f: D(f) \subseteq X \to X$ is a generalized contraction, that is, for any $x \in D(f)$, there is $\alpha(x) < 1$ such that $\|fx - fy\| \le \alpha(x)\|x - y\|$ for all y in $D(f)$, Webb [Web-3] showed a theorem on strong convergence of a general scheme —— the iteration of Mann [Man]. For nonexpansive mappings there is no such a general theorem on strong convergence. There are some strong and weak convergence theorems for nonexpansive mappings for special schemes, see Bruck's survey paper [Bru] and the references therein.

CHAPTER THREE

NORMAL STRUCTURE COEFFICIENTS

The notion of normal structure has proved to be a very useful one in fixed point theory [GoeK-1], [Ki]. Various types of normal structure coefficients such as the usual normal structure coefficient $N(X)$, the weakly convergent sequence coefficient $WCS(X)$ and $D(X)$ have been well studied [Am], [Ben-3], [By-2], [Mal], [Pr]. These coefficients have proved to be useful in order to obtain fixed points of nonexpansive mappings and uniformly Lipschitzian mappings [GoeK-1], [By-2], [CaM]. They are also helpful to compare k-set contractions and k-ball contractions [WebZ-2]. In this chapter, we will give several equivalent definitions of normal structure. Via these new characterizations of normal structure, we will define some other normal structure coefficients. We will study these geometrical numbers and use them to connect various measures of noncompactness. Also some relationships between them and $N(X)$ and $WCS(X)$ are given.

This chapter includes part of the work of [Z-2].

3.1. Various equivalent definitions of normal structure

Let C be a nonempty, closed, bounded and convex subset of a Banach space X. We will give the definitions of the m-Chebyshev radius of C and a ball measure of noncompactness of C, which allow us to give other characterizations of normal structure. By using the m-Chebyshev radius and this kind of ball measure, we can define several normal structure coefficients.

Definition 3.1.1. Let X be a Banach space and let C be a nonempty, closed, bounded and convex subset of X. For a fixed $m \in \mathbb{N}$, the m-Chebyshev radius of C is defined as:

$$r_m(C) := \inf\left\{ d > 0 : \text{there are } x_1, \ldots, x_m \in C \text{ so that } C \subseteq \bigcup_{i=1}^{m} B(x_i, d) \right\}$$

Also we define a ball measure of noncompactness $\beta_C(C)$ as:

$$\beta_C(C) := \inf\left\{ 2r > 0 : \text{there are finitely many } x_i \in C,\ i = 1, \ldots, n, \text{ such that } C \subseteq \bigcup_{i=1}^{n} B(x_i, r) \right\}.$$

Now we give some simple properties of $r_m(C)$ and $\beta_C(C)$.

Proposition 3.1.2. *Let X be a Banach space and let C be a closed, bounded and convex subset of X with* $\mathrm{diam}(C) > 0$. *Then*

1) $r_1(C) = r(C, C)$; *for any* $m \in \mathbb{N}$, $r_m(C) \geq r_{m+1}(C)$, $r_m(C) \geq \beta_C(C)/2$;

2) $\lim_{m \to \infty} r_m(C) = \beta_C(C)/2$;

3) $\beta_C(C) = 0$ *if and only if C is compact; in particular, if X is finite-dimensional,* $\beta_C(C) = 0$ *for any C;*

4) $r_m(C) > 0$ *for any* $m \in \mathbb{N}$;

5) *Let Y be another Banach space such that X and Y are isomorphic. Then for every bicontinuous linear operator U from X onto Y, we have* $r_m(UC) \leq \|U\| r_m(C)$ and $\beta_{UC}(UC) \leq \|U\| \beta_C(C)$. *In particular, for any* $a \in \mathbb{R} \setminus \{0\}$, $r_m(aC) = |a| r_m(C)$, $\beta_{aC}(aC) = |a| \beta_C(C)$. *Also for any* $x \in X$, $r_m(x + C) = r_m(C)$, $\beta_{x+C}(x + C) = \beta_C(C)$.

Proof. Property 1) is obvious. Since $\{r_m(C)\}$ is decreasing and has a lower bound $\beta_C(C)/2$, $\lim_{m\to\infty} r_m(C)$ exists. To prove 2), $\lim_{m\to\infty} r_m(C)\geq\beta_C(C)/2$ is clear. Also for any $a>\beta_C(C)/2$, there are $x_i\in C$, $i=1,\ldots,k$, such that $C\subseteq\bigcup_{i=1}^{k} B(x_i, a)$. So for any $m\geq k$, $r_m(C)\leq r_k(C)\leq a$. Thus $\lim_{m\to\infty} r_m(C)\leq a$. By the arbitrariness of a, we obtain $\lim_{m\to\infty} r_m(C)\leq\beta_C(C)/2$.

Property 3) is obvious. We show property 4) next. Since $\mathrm{diam}(C)>0$, there are $x, y\in C$ with $\|x-y\|\geq\mathrm{diam}(C)/2$. If $r_m(C)=0$, for any $\varepsilon>0$, there is a covering of C by $\bigcup_{i=1}^{m} B(x_i(\varepsilon), \varepsilon)$, $x_i(\varepsilon)\in C$. This union must cover the line segment $[x, y]$, so $[x, y]\subseteq\bigcup B(x_i(\varepsilon), \varepsilon)$, where the union contains at most m elements. Therefore $\|x-y\|\leq$ sum of diameters of the balls $\leq 2m\varepsilon$. As $\varepsilon>0$ is arbitrary, this is a contradiction. This proves property 4).

Lastly we prove 5). Since the proof for $r_m(C)$ and $\beta_C(C)$ are similar, we only give the proof for $\beta_C(C)$. For any $d>\beta_C(C)/2$, there are $x_1, \ldots, x_n\in C$ such that $C\subseteq\bigcup_{i=1}^{n} B(x_i, d)$. Hence $UC\subseteq\bigcup_{i=1}^{n} B(Ux_i, \|U\|d)$. Note that UC is closed, bounded and convex set in Y with $\mathrm{diam}(UC)>0$. Therefore, $\beta_{UC}(UC)\leq\|U\|\beta_C(C)$ since d is arbitrary.

For any $a\in\mathbb{R}\backslash\{0\}$, $U(x)=ax$ $(x\in X)$ is an invertible bounded linear operator from X onto X with $\|U\|=|a|$, thus $\beta_{aC}(aC)\leq|a|\beta_C(C)$. Since a and C are arbitrary, we also have $\beta_C(C)=\beta_{\frac{1}{a}(aC)}(\frac{1}{a}(aC))\leq|\frac{1}{a}|\beta_{aC}(aC)$.

For any $x\in X$, $x+C\subseteq\bigcup_{i=1}^{n} B(x+x_i, d)$ if $C\subseteq\bigcup_{i=1}^{n} B(x_i, d)$ $(x_i\in C,\ d>\beta_C(C)/2)$, thus $\beta_{x+C}(x+C)\leq\beta_C(C)$. Also we have

$$\beta_C(C)=\beta_{-x+(x+C)}(-x+(x+C))\leq\beta_{x+C}(x+C)$$

since x and C are arbitrary.

Next we show that equivalent definitions of normal structure can be given by using $r_m(C)$ and $\beta_C(C)$. The proof uses the next Lemma (see [GoeK-1], Lemma 4.1) which gives a necessary and sufficient condition for a space not to have normal structure.

Lemma 3.1.3. *A Banach space X does not have normal structure if, and only if, there is a bounded sequence $\{x_n\}\subseteq X$ with $\operatorname{diam}\{x_n\}>0$ such that*

$$\lim_{n\to\infty} \operatorname{dist}(x_{n+1}, \operatorname{co}\{x_i\}_1^n)=\operatorname{diam}\{x_n\}.$$

Theorem 3.1.4. *Let X be a Banach space, then the following properties are equivalent:*

1) *X has normal structure;*

2) *For any closed, bounded, convex subset C of X with $\operatorname{diam}(C)>0$, there exists $m\in\mathbb{N}$ such that $r_m(C)<\operatorname{diam}(C)$;*

3) *For any C as in 2), $\beta_C(C)<2\operatorname{diam}(C)$;*

4) *For any bounded nonconstant sequence $\{x_n\}$, $\beta_D(D)<2\operatorname{diam}\{x_n\}$, where $D=\overline{\operatorname{co}}\{x_n\}$.*

Proof. We prove 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$ 1). Suppose X has normal structure, then we have $r(C, C)<\operatorname{diam}(C)$. But $r_m(C)\le r_1(C)=r(C, C)$, so $r_m(C)<\operatorname{diam}(C)$. This proves that 1) implies 2). Since $\beta_C(C)/2\le r_m(C)$, 2) implies 3). That 3) implies 4) is obvious. Now we finish the proof by proving 4) $\Rightarrow$ 1). Suppose X satisfies property 4). If X does not have normal structure, by Lemma 3.1.3, there exists a bounded sequence $\{x_n\}\subseteq X$ such that

$$\lim_{n\to\infty} \operatorname{dist}(x_{n+1}, \operatorname{co}\{x_i\}_1^n)=\operatorname{diam}\{x_n\}:=d>0\ .$$

Let $D=\overline{\operatorname{co}}\{x_n\}$, then $\beta_D(D):=2r<2\operatorname{diam}(D)=2d$. Let $\varepsilon>0$ be chosen so that $2\varepsilon+r<d$.

Then there are $y_j \in D$, $j=1,2,\ldots,p$, such that $D \subseteq \bigcup_{j=1}^{p} B(y_j, r+\varepsilon)$. Some $B(y_j, r+\varepsilon)$ contains a subsequence $\{x_{n_k}\}$ of $\{x_n\}$. For this y_j, there is $z_l \in \mathrm{co}\{x_i\}_1^l$ such that $\|z_l - y_j\| < \varepsilon$. Hence

$$\|x_{n_k} - z_l\| \leq \|x_{n_k} - y_j\| + \|z_l - y_j\| < (r+\varepsilon)+\varepsilon = r+2\varepsilon.$$

Therefore for $n_k > l$, $\mathrm{dist}(x_{n_k}, \mathrm{co}\{x_i\}_1^{n_k-1}) \leq \|x_{n_k} - z_l\| \leq r+2\varepsilon < d$, a contradiction.

Using Theorem 3.1.4, we can define some types of normal structure coefficients.

Definition 3.1.5. Let X be a Banach space. For fixed $m \in \mathbb{N}$, we define a variant of the normal structure coefficient by:

$$N_m(X) = \inf\Big\{\mathrm{diam}(C)/r_m(C) : C \text{ a closed, bounded and convex subset of } X \text{ with } \mathrm{diam}(C) > 0\Big\}.$$

If X is infinite-dimensional, we define other two normal structure coefficient $N_\beta(X)$ and $N_s(X)$ as:

$$N_\beta(X) = \inf\Big\{2\mathrm{diam}(C)/\beta_C(C) : C \text{ a closed, bounded, convex and noncompact subset of } X\Big\}$$

and

$$N_s(X) = \inf\Big\{2\mathrm{diam}\{x_n\}/\beta_D(D) : \{x_n\} \text{ a bounded and noncompact sequence of } X,\ D = \overline{\mathrm{co}}\{x_n\}\Big\}.$$

Obviously, for any fixed $m \in \mathbb{N}$, $1 \leq N(X) = N_1(X) \leq N_m(X) \leq N_{m+1}(X) < +\infty$, and if X is infinite-dimensional, $1 \leq N(X) \leq N_m(X) \leq N_\beta(X) \leq N_s(X) \leq 2$. If F is a closed subspace of X, we have $N_m(F) \geq N_m(X)$, $N_\beta(F) \geq N_\beta(X)$ and $N_s(F) \geq N_s(X)$. In section 3.2, we will see that $N_\beta(X) = N_s(X)$, so later we just consider $N_m(X)$ and $N_\beta(X)$.

We know that $1 \leq N(X) \leq 2$ for any Banach space X, but the values of $N_m(X)$ $(m \geq 2)$ for finite-dimensional Banach spaces need not lie in the interval $[1, 2]$. To show this, we consider the real number set $\mathbb{R}$. It is easy to see that $N_m(\mathbb{R}) = 2m$. In fact, for any closed, bounded and convex subset C in $\mathbb{R}$ with $\mathrm{diam}(C) > 0$, $C = [a, b]$ $(a < b)$. So $\mathrm{diam}(C) = b - a$ and $r_m(C) = (b-a)/2m$.

Next we show that $\lim_{m \to \infty} N_m(X) = +\infty$ for any finite-dimensional Banach space X.

Proposition 3.1.6. For any finite-dimensional Banach space X, $N_m(X) \leq 2m$ and $\lim_{m \to \infty} N_m(X) = +\infty$.

Proof. For any $x_0 \in X$ with $x_0 \neq 0$, let $C = \{\alpha x_0 : 0 \leq \alpha \leq 1\}$, then we have

$$N_m(X) \leq \mathrm{diam}(C)/r_m(C) = 2m.$$

For any closed, bounded and convex subset C of X with $\mathrm{diam}(C) > 0$, let $d = \mathrm{diam}(C)$. For any $a > r_m(\overline{\mathrm{B}}_X)$, there are $x_1, \ldots, x_m \in \overline{\mathrm{B}}_X$ such that $\overline{\mathrm{B}}_X \subseteq \bigcup_{i=1}^{m} \mathrm{B}(x_i, a)$. Fix a point $z \in C$, we have

$$C \subseteq z + d\,\overline{\mathrm{B}}_X \subseteq \bigcup_{i=1}^{m} \left(z + d\,\mathrm{B}(x_i, a)\right) = \bigcup_{i=1}^{m} \mathrm{B}(z + dx_i, ad).$$

For any $1 \leq i \leq m$, if $C \cap \mathrm{B}(z + dx_i, ad) \neq \emptyset$, let $y_i \in C \cap \mathrm{B}(z + dx_i, ad)$; if $C \cap \mathrm{B}(z + dx_i, ad) = \emptyset$, let $y_i \in C$. Then $C = \bigcup_{i=1}^{m} \left(C \cap \mathrm{B}(z + dx_i, ad)\right) \subseteq \bigcup_{i=1}^{m} \mathrm{B}(y_i, 2ad)$. Hence $r_m(C) \leq 2ad$. We obtain $\mathrm{diam}(C)/r_m(C) \geq 1/2r_m(\overline{\mathrm{B}}_X)$ since a is arbitrary. Therefore $N_m(X) \geq 1/2r_m(\overline{\mathrm{B}}_X)$. Since $\overline{\mathrm{B}}_X$ is compact, we have

$$\lim_{m \to \infty} r_m(\overline{\mathrm{B}}_X) = \beta_{\overline{\mathrm{B}}_X}(\overline{\mathrm{B}}_X) = 0.$$

Hence $\lim_{m \to \infty} N_m(X) = +\infty$ is proved.

Property 5) in Proposition 3.1.2 allows us to give new expressions for

$N_m(X)$ and $N_\beta(X)$.

Proposition 3.1.7. *For a Banach space* X,

$$N_m(X)=\inf\left\{1/r_m(C)\colon\ C\subseteq X \text{ is closed and convex with } 0\in C \text{ and } \mathrm{diam}(C)=1\right\}.$$

If X *is infinite-dimensional, we have*

$$N_\beta(X)=\inf\left\{2/\beta_C(C)\colon\ C\subseteq X \text{ is closed, convex and noncompact with } 0\in C \text{ and } \mathrm{diam}(C)=1\right\}.$$

Bynum [By-1] gave a class of spaces which are reflexive but lack normal structure. Also a nonreflexive Banach space can have normal structure (cf. [GoeK-1]). Hence reflexivity and normal structure do not imply each other. However Maluta [Mal] proved that uniformly normal structure ($N(X)>1$ or $D(X)<1$) implies reflexivity. Our next result show that if $N_m(X)>1$ or $N_\beta(X)>1$, X is reflexive.

Theorem 3.1.8. *If* X *is a nonreflexive Banach space, then* $N_\beta(X)=N_m(X)=1$, $m\in\mathbb{N}$.

In order to prove Theorem 3.1.8, we need the following Theorem in [MiM], the Corollary of Theorem 2.

Theorem 3.1.9. (D. P. Mil'man and V. D. Mil'man) *If* X *is a nonreflexive Banach space, then for any* $\varepsilon>0$, *there is a sequence* $\{x_n\}\subseteq X$ *with* $\|x_n\|=1$ *such that* $1-\varepsilon\le\|x_{1n}-x_{n\omega}\|\le 1+\varepsilon$ *for any* $x_{1n}\in\mathrm{co}\{x_i\}_1^n$ *and* $x_{n\omega}\in\mathrm{co}\{x_i\}_{n+1}^\infty$.

Proof of Theorem 3.1.8. Since X is nonreflexive, X is infinite-dimensional. Then we have $1\le N_m(X)\le N_\beta(X)$. We finish the proof by showing $N_\beta(X)\le 1$. By Theorem 3.1.9, for any ε: $0<\varepsilon<1/4$, there is a sequence $\{x_n\}\subseteq X$ such that for any

$x_{1n} \in \mathrm{co}\{x_i\}_1^n$ and $x_{n\omega} \in \mathrm{co}\{x_i\}_{n+1}^{\infty}$, one has $1-\varepsilon \le \|x_{1n} - x_{n\omega}\| \le 1+\varepsilon$. Let $C = \overline{\mathrm{co}}\{x_n\}$, then $\mathrm{diam}(C) = \mathrm{diam}\{x_n\} \le 1+\varepsilon$. We claim that $\beta_C(C) \ge 2(1-3\varepsilon)$. If $\beta_C(C) < 2(1-3\varepsilon)$, there are $y_j \in C$, $j=1, \dots, k$, such that $C \subseteq \bigcup_{j=1}^{k} B(y_j, 1-3\varepsilon)$. For each y_j $(1 \le j \le k)$, there is $z_j \in \mathrm{co}\{x_i\}_1^{P_j}$ such that $\|z_j - y_j\| < \varepsilon$, and then $C \subseteq \bigcup_{j=1}^{k} B(z_j, 1-2\varepsilon)$. However if $N = \max\{P_j : 1 \le j \le k\}$, for any $j : 1 \le j \le k$, we have $\|x_{N+1} - z_j\| \ge 1-\varepsilon$, since $z_j \in \mathrm{co}\{x_i\}_1^N$ and $x_{N+1} \in \mathrm{co}\{x_i\}_{N+1}^{\infty}$. This is a contradiction. Hence

$$N_\beta(X) \le 2\mathrm{diam}(C)/\beta_C(C) \le 2(1+\varepsilon)/2(1-3\varepsilon) = (1+\varepsilon)/(1-3\varepsilon).$$

As this is true for every ε, $N_\beta(X) \le 1$.

Next we will give another property of $N_m(X)$ and $N_\beta(X)$ by using the Banach-Mazur distance. This property is similar to that of $N(X)$ and $WCS(X)$ given by Bynum in [By-2]. Let X and Y be isomorphic Banach space, if the values of $N_m(X)$, $N_\beta(X)$ and $d(X, Y)$ are known, the values of $N_m(Y)$ and $N_\beta(Y)$ can be estimated by using this result (see example 3.3.7 in section 3).

<u>Theorem 3.1.10.</u> *If X and Y are isomorphic Banach spaces, then we have*

$$N_m(X) \le d(X, Y) N_m(Y) \quad \textit{and} \quad N_\beta(X) \le d(X, Y) N_\beta(Y).$$

<u>Proof.</u> Let $C \subseteq Y$ be any closed, bounded, convex and noncompact set (or with $\mathrm{diam}(C) > 0$). Let $U: Y \to X$ be any isomorphism, then UC is a subset of X with the same properties as those of C. Thus, by 5) of Proposition 3.1.2 and the definitions of $N_m(X)$ and $N_\beta(X)$, we have

$$\begin{aligned} r_m(C) = r_m(U^{-1}U(C)) &\le \|U^{-1}\| r_m(UC) \\ &\le \|U^{-1}\| \,\mathrm{diam}(UC)/N_m(X) \\ &\le \|U^{-1}\| \|U\| \,\mathrm{diam}(C)/N_m(X); \end{aligned}$$

and

$$\begin{aligned}\beta_C(C) = \beta_{U^{-1}U(C)}(U^{-1}U(C)) &\le \|U^{-1}\| \beta_{UC}(UC)\\ &\le \|U^{-1}\| 2\mathrm{diam}(UC)/N_\beta(X)\\ &\le \|U^{-1}\| \|U\| 2\mathrm{diam}(C)/N_\beta(X)\end{aligned}$$

Then $N_m(X) \le \|U^{-1}\| \|U\| \mathrm{diam}(C)/r_m(C)$ and $N_\beta(X) \le \|U^{-1}\| \|U\| 2\mathrm{diam}(C)/\beta_C(C)$. By the arbitrariness of U, we have $N_m(X) \le d(X, Y)\,\mathrm{diam}(C)/r_m(C)$ and $N_\beta(X) \le d(X, Y) 2\mathrm{diam}(C)/\beta_C(C)$. The result follows since C is arbitrary.

3.2. Some properties of $\beta_C(\Omega)$ and $N_\beta(X)$

Let X be an infinite-dimensional Banach space. We will use $N_\beta(X)$ to connect various measures of noncompactness. First we give the definition of a ball measure. Let C be a nonempty, closed and convex set in X, for any bounded subset Ω of C, $\beta_C(\Omega)$ is defined by:

$$\beta_C(\Omega) := \inf\Big\{2r > 0 : \text{there are finitely many } x_i \in C,\ i = 1, 2, \ldots, n, \text{ such that } \Omega \subseteq \bigcup_{i=1}^{n} B(x_i, r)\Big\}.$$

Next we give some properties of this variant of the ball measure.

Lemma 3.2.1. *Let X be a Banach space, C a nonempty, closed and convex subset of X. Then for any bounded subsets A and B of C, we have:*

1) $\beta_C(A) = 0$ *if and only if A is precompact;*

2) *If $A \subseteq B$, then* $\beta_C(A) \le \beta_C(B)$;

3) $\beta_C(\overline{A}) = \beta_C(A)$;

4) $\beta_C(A \cup B) = \max\{\beta_C(A), \beta_C(B)\}$;

5) *For any λ:* $0 < \lambda < 1$, $\beta_C(\lambda A + (1-\lambda)B) \le \lambda\beta_C(A) + (1-\lambda)\beta_C(B)$;

6) $\beta_C(\mathrm{co}A)=\beta_C(A)$.

Proof. The properties 1)–4) are true since we can view C as a metric space (Lemma 1.3.3). For 5). Since $A\subseteq C$, $B\subseteq C$, and C is convex, $\lambda A+(1-\lambda)B\subseteq C$. Put $\beta_C(A)=2a$, and $\beta_C(B)=2b$. For any $\varepsilon>0$, there are $x_i\in C$, $i=1,\ldots,n$, and $y_j\in C$, $j=1,\ldots,k$, such that $A\subseteq \bigcup_{i=1}^{n} \mathrm{B}(x_i, a+\varepsilon)$ and $B\subseteq \bigcup_{j=1}^{k} \mathrm{B}(y_j, b+\varepsilon)$. Then

$$\lambda A+(1-\lambda)B\subseteq \bigcup_{i=1}^{n}\bigcup_{j=1}^{k} \mathrm{B}(\lambda x_i+(1-\lambda)y_j, \lambda a+(1-\lambda)b+\varepsilon),$$

since $\lambda x_i+(1-\lambda)y_j\in C$, 5) follows by the arbitrariness of ε.

We show 6) next. For any $d>\beta_C(A)/2$, there are $x_1,\ldots,x_n\in C$ such that $A\subseteq \bigcup_{i=1}^{n} \mathrm{B}(x_i, d)$. We claim $\mathrm{co}A\subseteq \bigcup_{z\in \mathrm{co}\{x_i\}_1^n} \mathrm{B}(z, d)$. In fact, for any $y\in\mathrm{co}A$, $y=\sum_{j=1}^{k}\lambda_j y_j$ with $\lambda_j\geq 0$, $\sum_{j=1}^{k}\lambda_j=1$ and $y_j\in A$. For each y_j, there is x_{n_j} $(1\leq n_j\leq n)$ such that $\|y_j-x_{n_j}\|\leq d$, thus $\|y-\sum_{j=1}^{k}\lambda_j x_{n_j}\|\leq \sum_{j=1}^{k}\lambda_j\|y_j-x_{n_j}\|\leq d$. Note that $\sum_{j=1}^{k}\lambda_j x_{n_j}\in\mathrm{co}\{x_i\}_1^n$, so this is the claimed result. Since $\mathrm{co}\{x_i\}_1^n$ is compact, for any $\varepsilon>0$, there exist $z_1,\ldots,z_p\in\mathrm{co}\{x_i\}_1^n$ such that if $z\in\mathrm{co}\{x_i\}_1^n$, then for some z_l $(1\leq l\leq p)$, $\|z-z_l\|<\varepsilon$. Therefore, $\mathrm{co}A\subseteq \bigcup_{l=1}^{p} \mathrm{B}(z_l, d+\varepsilon)$. Since $z_l\in\mathrm{co}\{x_i\}_1^n\subseteq C$, $1\leq l\leq p$, we obtain that $\beta_C(\mathrm{co}A)\leq 2d$. Then $\beta_C(\mathrm{co}A)\leq\beta_C(A)$ by the arbitrariness of d. 6) follows by using 2).

We establish a connection between ball and set measures of noncompactness by using $N_\beta(X)$.

Theorem 3.2.2. *Let X be an infinite-dimensional Banach space, for any closed, bounded, convex subset C of X, we have $\beta_C(C)\leq 2\alpha(C)/N_\beta(X)$. For any bounded* *subset Ω of X, we have $\beta(\Omega)\leq 2\alpha(\Omega)/N_\beta(X)$.*

Proof. Suppose C is noncompact. For any $a>\alpha(C)$, there are $C_i\subseteq C$, $i=1,\ldots,n$, such that $C=\bigcup_{i=1}^{n} C_i$ and $\mathrm{diam}(C_i)\leq a$ for $1\leq i\leq n$. There is a k: $1\leq k\leq n$ such that $\beta_C(C_k)=\max\{\beta_C(C_i):\ 1\leq i\leq n\}$. Then

$$\beta_C(C)=\beta_C(C_k)=\beta_C(\overline{\mathrm{co}}C_k)\leq\beta_{\overline{\mathrm{co}}C_k}(\overline{\mathrm{co}}C_k)$$
$$\leq 2\mathrm{diam}(\overline{\mathrm{co}}C_k)/N_\beta(X)=2\mathrm{diam}(C_k)/N_\beta(X)\leq 2a/N_\beta(X).$$

By the arbitrariness of a, $\beta_C(C)\leq 2\alpha(C)/N_\beta(X)$. Since $\beta(\Omega)=\beta(\overline{\mathrm{co}}\Omega)\leq\beta_{\overline{\mathrm{co}}\Omega}(\overline{\mathrm{co}}\Omega)$, and $\alpha(\Omega)=\alpha(\overline{\mathrm{co}}\Omega)$, $\beta(\Omega)\leq 2\alpha(\Omega)/N_\beta(X)$ is true for any bounded subset Ω of X.

Next we show two Lemmas which allow the separation measure δ to replace the set measure α in Theorem 3.2.2. Our first Lemma gives an equivalent definition of $\delta(\Omega)$, which is also shown in [BenL-2] by Benavides and Lopez Acedo.

Lemma 3.2.3. *If Ω is an infinite bounded set in a metric space M, then*

$$\mu(\Omega):=\sup\{\alpha(\Omega'):\ \Omega'\text{ is an }\alpha\text{-minimal subset of }\Omega\}=\delta(\Omega).$$

Proof. Let Ω' be an α-minimal subset of Ω (By Proposition 1.3.7, such Ω' exists). For any $\varepsilon>0$, there is an infinite subset A of Ω' such that $d(x,y)>\alpha(\Omega')-\varepsilon$ for any $x,y\in A$ (Lemma 1.3.9), that is $\delta(\Omega)\geq\alpha(\Omega')-\varepsilon$. Then $\delta(\Omega)\geq\alpha(\Omega')$ by letting $\varepsilon\to 0$, so $\delta(\Omega)\geq\mu(\Omega)$ as Ω' is arbitrary.

Conversely, for any $\varepsilon>0$, there exists an infinite subset A of Ω such that $d(x,y)\geq\delta(\Omega)-\varepsilon$ for all $x,y\in A$, $x\neq y$. Let A' be an α-minimal subset of A. Then we have $\alpha(A')\geq\delta(\Omega)-\varepsilon$ since $d(x',y')\geq\delta(\Omega)-\varepsilon$ for any x', $y'\in A'$, $x'\neq y'$. So $\mu(\Omega)\geq\delta(\Omega)-\varepsilon$, and the result follows since ε is arbitrary.

Lemma 3.2.4. *Let X be an infinite-dimensional Banach space, C a closed, bounded, convex subset of X with $\beta_C(C)=2a>0$. Then for any r: $0<r<a$, there is a sequence $\{x_n\}\subseteq C$ such that $\beta_D(D)\geq 2r$, where $D=\overline{\mathrm{co}}\{x_n\}$.*

Proof. We construct a sequence $\{x_n\}$ satisfying the conclusion of the lemma. Let x_1 be any point in C. Suppose $x_1, \ldots, x_n$ have been obtained and let $D_n=\mathrm{co}\{x_i\}_1^n$. We claim that there exists $x_{n+1}\in C$ such that $\mathrm{dist}(x_{n+1}, D_n)>r$. In fact, otherwise, $\mathrm{dist}(x, D_n)\leq r$ for all $x\in C$. Since D_n is precompact, then $\beta_C(D_n)=0$. For any $\varepsilon>0$, there are $y_1, \ldots, y_m\in C$ such that $D_n\subseteq \bigcup_{j=1}^{m} \mathrm{B}(y_j, \varepsilon)$. Then $C\subseteq \bigcup_{j=1}^{m} \mathrm{B}(y_j, r+\varepsilon)$. Hence $\beta_C(C)\leq 2(r+\varepsilon)$. Since ε is arbitrary, this gives $\beta_C(C)\leq 2r<2a$, a contradiction.

By induction, we obtain a sequence $\{x_n\}\subseteq C$ satisfying $\mathrm{dist}(x_{n+1}, D_n)>r$ for all $n=1, 2, 3, \ldots,$ where $D_n=\mathrm{co}\{x_i\}_1^n$. We prove $\beta_D(D)=\beta_D(\{x_n\})\geq 2r$, where $D=\overline{\mathrm{co}}\{x_n\}$. If $\beta_D(\{x_n\})=2b<2r$, let $\delta>0$ be such that $b+2\delta<r$. Then there are $y_j\in D$, $j=1, 2, \ldots, l$, such that $\{x_n\}\subseteq \bigcup_{j=1}^{l} \mathrm{B}(y_j, b+\delta)$. For every $y_j\in D$, there is a $w_j\in D_{p_j}$ such that $\|y_j-w_j\|\leq\delta$, hence $\{x_n\}\subseteq \bigcup_{j=1}^{l} \mathrm{B}(w_j, b+2\delta)$. Let $N=\max\{p_j : 1\leq j\leq l\}$. Then for all $j=1, 2, \ldots, l$, $w_j\in D_{\mathrm{N}}$, so $\|x_{\mathrm{N}+1}-w_j\|\geq\mathrm{dist}(x_{\mathrm{N}+1}, D_{\mathrm{N}})>r>b+2\delta$ for all $j : 1\leq j\leq l$, that is, $x_{\mathrm{N}+1}\notin \bigcup_{j=1}^{l} \mathrm{B}(w_j, b+2\delta)$. This is a contradiction.

Theorem 3.2.5. *Let X be an infinite-dimensional Banach space. For any closed, bounded, convex subset C of X, we have $\beta_C(C)\leq 2\delta(C)/N_\beta(X)$. For any bounded subset Ω of X, we have $\beta(\Omega)\leq 2\delta(\Omega)/N_\beta(X)$.*

Proof. If $\beta_C(C)=2a\neq 0$, for any ε: $0<\varepsilon<a$, by Lemma 3.2.4, there is a sequence $\{x_n\}\subseteq C$ such that $\beta_D(D)\geq 2(a-\varepsilon)$, where $D=\overline{\mathrm{co}}\{x_n\}$. Using Proposition 1.3.8, there

is a β-minimal subset D' of D such that $\beta_D(D')=\beta_D(D)$. For D', there exists an α-minimal subset $D''\subseteq D'$ (Proposition 1.3.7). Let $\{y_n\}$ be a sequence in D'', then $\{y_n\}$ is α-minimal and β-minimal, and $\beta_D(\{y_n\})=\beta_D(D')=\beta_D(D)$. Thus

$$\begin{aligned}\beta_C(C)&\le\beta_D(D)+2\varepsilon=\beta_D(\{y_n\})+2\varepsilon=\beta_D(\overline{\mathrm{co}}\{y_n\})+2\varepsilon\\&\le\beta_{\overline{\mathrm{co}}\{y_n\}}(\overline{\mathrm{co}}\{y_n\})+2\varepsilon\le2\alpha(\overline{\mathrm{co}}\{y_n\})/N_\beta(\overline{\mathrm{span}}\{x_n\})+2\varepsilon\\&\le2\alpha(\{y_n\})/N_\beta(X)+2\varepsilon\le2\delta(C)/N_\beta(X)+2\varepsilon,\end{aligned}$$

the last inequality used Lemma 3.2.3. We have $\beta_C(C)\le2\delta(C)/N_\beta(X)$ as ε is arbitrary. Since $\beta(\Omega)=\beta(\overline{\mathrm{co}}\Omega)\le\beta_{\overline{\mathrm{co}}\Omega}(\overline{\mathrm{co}}\Omega)$, and $\delta(\Omega)=\delta(\overline{\mathrm{co}}\Omega)$, $\beta(\Omega)\le2\delta(\Omega)/N_\beta(X)$ is true for any bounded subset Ω of X.

From Theorems 3.2.2 and 3.2.5, we can give other expressions for $N_\beta(X)$.

Proposition 3.2.6. *For an infinite-dimensional Banach space X,*

$$N_\beta(X)=\inf\Big\{2\alpha(C)/\beta_C(C)\colon\ C\ \textit{a closed, bounded, convex and noncompact subset of}\ X\Big\}$$

and

$$N_\beta(X)=\inf\Big\{2\delta(C)/\beta_C(C)\colon\ C\ \textit{a closed, bounded, convex and noncompact subset of}\ X\Big\}.$$

By using Lemma 3.2.4, we can prove $N_\beta(X)=N_s(X)$ for Banach space X. In fact, for any $0<\varepsilon<1$, there is closed, bounded, convex and noncompact subset C in X so that

$$N_\beta(X)\ge2\mathrm{diam}(C)/\beta_C(C)-\varepsilon.$$

Let $\beta_C(C)=2a$, by Lemma 3.2.4, there is a sequence $\{x_n\}\subseteq C$ such that $\beta_D(D)\ge2a(1-\varepsilon)$ with $D=\overline{\mathrm{co}}\{x_n\}$. Then we have

$$N_\beta(X)\ge2\mathrm{diam}\{x_n\}(1-\varepsilon)/\beta_D(D)-\varepsilon\ge N_s(X)(1-\varepsilon)-\varepsilon.$$

Hence $N_\beta(X) \geq N_s(X)$ since ε is arbitrary.

3.3. $N_\beta(X)$ and $WCS(X)$ are equal in reflexive Banach space

In this section, we will show that for any infinite-dimensional reflexive Banach space X, $N_\beta(X) = WCS(X)$. To prove $N_\beta(X) \geq WCS(X)$, the next result is the essential one needed.

Theorem 3.3.1. *Let X be an infinite-dimensional reflexive Banach space, then for any closed, bounded, convex subset C of X, $\beta_C(C) \leq 2\delta(C)/WCS(X)$.*

In order to prove Theorem 3.3.1, we need the following useful fact in [Re] proved by Reich by a diagonalization argument.

Lemma 3.3.2. *Let $\{x_n\}$ be a bounded sequence contained in a separable subset D of a Banach space X. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} \|x_{n_k} - x\|$ exists for all $x \in D$.*

Proof of Theorem 3.3.1. First we consider a separable space X. Suppose $\beta_C(C) \neq 0$. As C is separable, there is a β-minimal subset C_1 of C such that $\beta_C(C) = \beta_C(C_1)$ (Proposition 1.3.8). There exists an α-minimal subset $C_2 \subseteq C_1$ (Proposition 1.3.7). For C_2, since X is reflexive and separable, for any $\varepsilon > 0$, by taking subsequences several times, we can obtain a sequence $\{x_n\} \subseteq C_2$ such that:

1) $x_n \neq x_m$ if $m \neq n$;

2) $\alpha(C_2) \geq \operatorname{diam}\{x_n\} - \varepsilon$ (Lemma 1.3.9);

3) $\{x_n\}$ converges weakly to $x \in C$;

4) For any $z \in X$, $\lim_{n\to\infty} \|x_n - z\|$ exists (Lemma 3.3.2).

Then

$$\beta_C(C)=\beta_C(\{x_n\}) \le 2 \inf_{z \in \text{co}\{x_n\}} \lim_{n\to\infty} \|x_n - z\| \le 2\text{diam}_a\{x_n\}/WCS(X)$$

$$\le 2\text{diam}\{x_n\}/WCS(X) \le 2(\alpha(C_2)+\varepsilon)/WCS(X) \le 2(\delta(C)+\varepsilon)/WCS(X).$$

So the result is true for a separable reflexive Banach space.

Now suppose X is not separable. Let $\beta_C(C)=2a \neq 0$. For any $\varepsilon > 0$, from Lemma 3.2.4, there is a sequence $\{x_n\} \subseteq C$ such that $\beta_D(D) \ge 2(a-\varepsilon)$, where $D=\overline{\text{co}}\{x_n\}$. Then

$$\beta_C(C) \le \beta_D(D)+2\varepsilon \le 2\delta(D)/WCS(\overline{\text{span}}\{x_n\})+2\varepsilon \le 2\delta(C)/WCS(X)+2\varepsilon.$$

Note that $\overline{\text{span}}(\{x_n\})$ is a separable reflexive closed subspace of X, so it is reflexive. We also used $WCS(\overline{\text{span}}\{x_n\}) \ge WCS(X)$ in the above inequalities. The result follows since $\varepsilon > 0$ is arbitrary.

From Theorem 3.3.1 and Proposition 3.2.6, it follows that $WCS(X) \le N_\beta(X)$ in any reflexive Banach space. In fact equality holds. In order to prove this equality, we need the following useful characterization of $WCS(X)$ given by Prus **[Pr]**.

Lemma 3.3.3. *Let X be an infinite-dimensional reflexive Banach space, then $WCS(X)$ has the following equivalent expression*

$$WCS(X)=\inf\left\{\text{diam}\{x_n\}:\ \|x_n\|=1,\ x_n \text{ converges weakly to } 0\right\}.$$

Theorem 3.3.4. *If X is an infinite-dimensional reflexive Banach space, then $WCS(X)=N_\beta(X)$.*

Proof. We only need to prove $WCS(X) \geq N_\beta(X)$. By Lemma 3.3.3, for any $\varepsilon > 0$, there is a sequence $\{x_n\}$, with $\|x_n\| = 1$ and x_n converges weakly to 0, such that $WCS(X) > \operatorname{diam}\{x_n\} - \varepsilon$. Without loss of generality, we can suppose that $\lim_{n\to\infty}\|x_n - z\|$ exists for all $z \in \overline{\mathrm{co}}\{x_n\}$ (otherwise, using Lemma 3.3.2 take a subsequence).

Let $A_k = \overline{\mathrm{co}}\{x_n\}_{n=k}^{\infty}$, then $A := \bigcap_{k=1}^{\infty} A_k = \{0\}$. Although this is a result in [BenL-2], we give a proof here. Since $0 \in A_k$ for each k, $0 \in A$. For any $x \in A$, there is a $x^* \in X^*$ with $\|x^*\| = 1$ so that $x^*(x) = \|x\|$. Since $w\text{-}\lim_{n\to\infty} x_n = 0$, for any $\varepsilon > 0$, there is a $K \in \mathbb{N}$ so that $|x^*(x_n)| < \varepsilon$ whenever $n \geq K$. For any $k \geq K$, since $x \in A_k$, there are λ_j, $j = k, \ldots, m(k)$, such that $\lambda_j \geq 0$ and $\sum_{j=k}^{m(k)} \lambda_j = 1$ and $\|x - \sum_{j=k}^{m(k)} \lambda_j x_j\| < \varepsilon$. But

$$\|x - \sum_{j=k}^{m(k)} \lambda_j x_j\| \geq x^*(x - \sum_{j=k}^{m(k)} \lambda_j x_j) = \|x\| - \sum_{j=k}^{m(k)} \lambda_j x^*(x_j) \geq \|x\| - \sum_{j=k}^{m(k)} \lambda_j \varepsilon = \|x\| - \varepsilon,$$

we have $\|x\| < 2\varepsilon$. Thus $x = 0$ since ε is arbitrary.

Next we claim that

$$\lim_{k\to\infty} \Big\{ \inf_{z \in A_k} \lim_{n\to\infty} \|x_n - z\| \Big\} = \lim_{n\to\infty} \|x_n - 0\| = 1$$

Let $a_k = \inf_{z \in A_k} \lim_{n\to\infty} \|x_n - z\|$, then $\{a_k\}$ is an increasing sequence and $a_k \leq 2$. So the limit of the left hand side exists. $\lim_{k\to\infty} a_k \leq \lim_{n\to\infty} \|x_n - 0\|$ since $0 \in A_k$ for each k. For any $\varepsilon > 0$, there is $z_k \in A_k$ so that $a_k > \lim_{n\to\infty} \|x_n - z_k\| - \varepsilon$. There exists a subsequence $\{z_{k_j}\}$ of $\{z_k\}$ so that $\lim_{j\to\infty} \lim_{n\to\infty} \|x_n - z_{k_j}\|$ exists and $\{z_{k_j}\}$ converges weakly to some z_0. For any $i \in \mathbb{N}$, since $z_{k_j} \in A_i$ if $k_j \geq i$. so $z_0 \in A_i$. Since $\bigcap_{i=1}^{\infty} A_i = \{0\}$, we have $z_0 = 0$. For each n, there is $x_n^* \in X^*$ with $\|x_n^*\| = 1$ and $x_n^*(x_n) = \|x_n\|$. Since $w\text{-}\lim_{j\to\infty} z_{k_j} = 0$, there is a $K(n) \in \mathbb{N}$ so that $|x_n^*(z_{k_j})| < 1/2^n$ whenever $j > K(n)$. So if $j > K(n)$, we have

$$\|x_n-z_{k_j}\|\geq x_n^*(x_n-z_{k_j})=\|x_n\|-x_n^*(z_{k_j})\geq\|x_n\|-1/2^n.$$

Then

$$\lim_{j\to\infty}\lim_{n\to\infty}\|x_n-z_{k_j}\|\geq\lim_{n\to\infty}\|x_n\|.$$

Therefore

$$\lim_{k\to\infty}a_k=\lim_{j\to\infty}a_{k_j}\geq\lim_{j\to\infty}\lim_{n\to\infty}\|x_n-z_{k_j}\|-\varepsilon\geq\lim_{n\to\infty}\|x_n\|-\varepsilon.$$

which proves $\lim_{k\to\infty}a_k\geq\lim_{n\to\infty}\|x_n\|$ since ε is arbitrary.

Now we see that there is a $K\in\mathbb{N}$ so that $\inf_{z\in A_K}\lim_{n\to\infty}\|x_n-z\|>1-\varepsilon$. Hence $\beta_{A_K}(A_K)\geq 2(1-\varepsilon)$. In fact, if $\beta_{A_K}(A_K)<2(1-\varepsilon)$, then there are $z_1,\ldots,z_m\in A_K$ such that $A_K\subseteq\bigcup_{i=1}^{m}B(z_i,1-\varepsilon)$. There is z_i $(1\leq i\leq m)$ such that the ball $B(z_i,1-\varepsilon)$ contains a subsequence $\{x_{n_j}\}$ of $\{x_n\}_{n=K}^{\infty}$, then

$$\inf_{z\in A_K}\lim_{n\to\infty}\|x_n-z\|\leq\lim_{n\to\infty}\|x_n-z_i\|=\lim_{j\to\infty}\|x_{n_j}-z_i\|\leq 1-\varepsilon.$$

Note that $\beta_{A_K}(A_K)/(\beta_{A_K}(A_K)+2\varepsilon)\geq 1-\varepsilon$ since $\beta_{A_K}(A_K)\geq 2(1-\varepsilon)$, and we have

$$WCS(X)>\operatorname{diam}(A_K)-\varepsilon\geq 2\operatorname{diam}(A_K)/(\beta_{A_K}(A_K)+2\varepsilon)-\varepsilon\geq N_\beta(X)(1-\varepsilon)-\varepsilon.$$

the result follows by the arbitrariness of ε.

<u>Remark</u>. The author originally proved $WCS(X)\geq N_\beta(X)$ for reflexive Opial spaces (in Chapter 4 we will give the definition of Opial space). I thank Professor T. D. Benavides (Personal communication) for his remarking that the result is true in every reflexive Banach space by using the ideas of [BenL-2]. The proof above is obtained by using his suggestion.

From Theorem 3.1.8, we see that if $N_m(X)>1$ or $N_\beta(X)>1$, X is reflexive and has normal structure. But the converse is not true. Let X be the l^2-direct sum of the sequence of spaces $\{l^m\}_{m\geq 2}$. Bynum [By-2] showed that X is reflexive and has normal structure, but $N(X)=WCS(X)=1$. Hence by Theorem 3.3.4, $N_\beta(X)=1$. Also $N_m(X)=1$ since $N_m(X)\leq N_\beta(X)$.

We know that $WCS(X)=1/D(X)$ for infinite-dimensional reflexive Banach spaces (see Chapter 1). By Theorems 3.1.8 and 3.3.4, noting that $D(X)=1$ if X is a nonreflexive Banach space, the following corollary is immediate.

Corollary 3.3.5. *For any infinite-dimensional Banach space,* $N_\beta(X)=1/D(X)$.

Also from Theorems 1.2.14, 1.2.16 and 1.2.17, and the remark for Theorem 2.1.8. as consequence of Theorem 3.3.4, we have

Corollary 3.3.6. *Let* $m\in\mathbb{N}$. *In an infinite-dimensional Hilbert space* H, *we have* $N_m(H)=N_\beta(H)=\sqrt{2}$. *For* $1<p<\infty$, *we have* $N_m(L^p)=N_\beta(L^p)=\min\{2^{1/p}, 2^{1-1/p}\}$ *and* $N_\beta(l^p)=2^{1/p}$. *For* $1<p<2$, $\min\{2^{1/p}, 2^{1-1/p}\}\leq N_m(l^p)\leq 2^{1/p}$; *and for* $p>2$, $N_m(l^p)=2^{1/p}$.

Remark. Since $N(l^p)=\min\{2^{1/p}, 2^{1-1/p}\}$, l^p $(1<p<2)$ spaces give examples where $N(l^p)<N_\beta(l^p)$. Let X be the l^2-direct sum of the spaces l_n^∞ $(n\geq 1)$, Baillon showed that $N(X)=1$ and $WCS(X)=\sqrt{2}$ [By-2]. Then $N_\beta(X)=\sqrt{2}$. Hence $N_\beta(X)>1$ does not characterize the usual uniformly normal structure.

We think that $N_m(X)$ and $N_\beta(X)$ are not equal for infinite-dimensional Banach space X, but we have not obtained such an example. Also we do not know if there is a space so that $N_1(X)<N_2(X)<N_3(X)< \ldots <N_\beta(X)$.

Using Theorem 3.1.10 and the known values for $N_m(X)$ and $N_\beta(X)$, we can give examples where $N_m(X)>1$ and $N_\beta(X)>1$.

Example 3.3.7. For the space $E_\lambda=(l^2, |.|_\lambda)$ $(1<\lambda<\sqrt{2})$ defined in Example 2.1.10, $d(l^2, E_\lambda)\le\lambda$. Thus by Theorem 3.1.10, we have $N_m(l^2)\le\lambda N_m(E_\lambda)$ and $N_\beta(l^2)\le\lambda N_\beta(E_\lambda)$. Therefore $N_m(E_\lambda)\ge\sqrt{2}/\lambda>1$ and $N_\beta(E_\lambda)\ge\sqrt{2}/\lambda>1$.

Let X be an infinite-dimensional Banach space, we consider another geometrical coefficient $B(X)$ of X which is defined by:

$$B(X):=\inf\Big\{\operatorname{diam}\{x_n\}/\inf_{z\in\overline{\mathrm{co}}\{x_n\}}\lim_{n\to\infty}\|x_n-z\| : \{x_n\} \text{ is bounded and not strongly convergent, } \lim_{n\to\infty}\|x_n-z\| \text{ exists for all } z\in\overline{\mathrm{co}}\{x_n\}\Big\}.$$

This coefficient is a modification of $BS(X)$ defined by Bynum [By-2]:

$$BS(X):=\inf\Big\{\operatorname{diam}_a\{x_n\}/\inf_{z\in\overline{\mathrm{co}}\{x_n\}}\limsup_{n\to\infty}\|x_n-z\| : \{x_n\} \text{ is bounded and not strongly convergent}\Big\}.$$

Obviously $1\le B(X)\le 2$. Lim [Lim-2] proved $BS(X)=N(X)$, but our next result shows that $B(X)=WCS(X)$ for any reflexive Banach space X.

Proposition 3.3.8. *Let X be an infinite-dimensional Banach space. If X is nonreflexive, then $B(X)=1$; if X is reflexive, then $B(X)=WCS(X)=N_\beta(X)$.*

Proof. First we prove that $B(X)\le N_\beta(X)$. By Proposition 3.1.7, for any $0<\varepsilon<1/2$, there is a closed, convex and noncompact subset C of X with $\operatorname{diam}(C)=1$ such that $N_\beta(X)\ge 2/\beta_C(C)-\varepsilon$. By Lemma 3.2.4, there is a sequence $\{x_n\}\subseteq C$ such that $\beta_D(D)\ge\beta_C(C)(1-\varepsilon)$, where $D=\overline{\mathrm{co}}\{x_n\}$. From Proposition 1.3.7, 1.3.8 and Lemma 3.3.2, by taking subsequences several times, we can obtain a subsequence $\{y_n\}$ (say) of $\{x_n\}$ such that $\{y_n\}$ is α-minimal and β-minimal, $\lim_{n\to\infty}\|y_n-z\|$ exists for all $z\in D$, and $\beta_D(D)=\beta_D(\{y_n\})\le 2\inf_{z\in\overline{co}\{y_n\}}\lim_{n\to\infty}\|y_n-z\|$. Noting that $\operatorname{diam}\{y_n\}\le 1$, we obtain

$$N_\beta(X)\ge 2(1-\varepsilon)/\beta_D(D)-\varepsilon\ge\left(\operatorname{diam}\{y_n\}/\inf_{z\in\overline{co}\{y_n\}}\lim_{n\to\infty}\|y_n-z\|\right)(1-\varepsilon)-\varepsilon\ge B(X)(1-\varepsilon)-\varepsilon.$$

By the arbitrariness of ε, we have $N_\beta(X)\ge B(X)$.

If X is nonreflexive, since $N_\beta(X)=1$, then $B(X)=1$. If X is reflexive, it suffices to prove $WCS(X)\le B(X)$ to obtain the equalities. For any $\varepsilon>0$, by the definition of $B(X)$, there is a bounded and not strongly convergent sequence $\{x_n\}\subseteq X$ such that $\lim_{n\to\infty}\|x_n-z\|$ exists for all $z\in\overline{co}\{x_n\}$ and

$$B(X)\ge\operatorname{diam}\{x_n\}/\inf_{z\in\overline{co}\{x_n\}}\lim_{n\to\infty}\|x_n-z\|-\varepsilon.$$

Since X is reflexive, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ is weakly convergent but not strongly convergent. Therefore

$$B(X)\ge\operatorname{diam}_a\{x_{n_k}\}/\inf_{z\in\overline{co}\{x_{n_k}\}}\lim_{n\to\infty}\|x_{n_k}-z\|-\varepsilon\ge WCS(X)-\varepsilon.$$

By the arbitrariness of ε, we obtain $B(X)\ge WCS(X)$.

CHAPTER FOUR

VARIOUS MEASURES OF NONCOMPACTNESS

Various measures of noncompactness of a bounded set in a metric space have been well studied in recent years. The set measure, the ball measure and the separation measure defined in chapter one have been of particular interest. There measures of noncompactness share similar properties but they are not equal. Generally, the evident inequalities $\alpha(\Omega)\leq\beta(\Omega)\leq 2\alpha(\Omega)$ and $\delta(\Omega)\leq\alpha(\Omega)\leq\beta(\Omega)\leq 2\delta(\Omega)$ are the best possible, as shown in Chapter one.

These measures are closely related to geometrical properties of the underlying space. It is possible to improve on the inequalities $\beta(\Omega)\leq 2\alpha(\Omega)$ and $\beta(\Omega)\leq 2\delta(\Omega)$ in certain spaces. Indeed, better inequalities are obtained in [Dan-1], [Dan-2] and [WebZ-2], see also Chapter three. In this chapter, we will study these measures and establish various connections between them using certain geometrical properties of the space. We will improve on $\beta(\Omega)\leq 2\alpha(\Omega)$, $\beta(\Omega)\leq 2\delta(\Omega)$ and $\delta(\Omega)\leq\beta(\Omega)$ by using some geometrical coefficients, which we study here.

The inequalities for $\alpha(\Omega)$, $\beta(\Omega)$ and $\delta(\Omega)$ have consequences for k-set contractions, k-ball contractions and k-δ-contractions. Sharper inequalities than $\alpha(f)/2\leq\beta(f)\leq 2\alpha(f)$ and $\delta(f)/2\leq\beta(f)\leq 2\delta(f)$ are obtained. We also show that $\delta(f)\leq\alpha(f)$. Our results include showing that if the underlying spaces are Hilbert or l^p spaces, $\beta(f)\leq\delta(f)\leq\alpha(f)$, so these recover the results of Benavides [Ben-1], [Ben-2].

This chapter includes part of the work of [WebZ-2] and [Z-1].

4.1. The measures of noncompactness in metric spaces

Let (M, ρ) be a metric space and let Ω be a bounded set in M. We will connect $\alpha(\Omega)$ (or $\delta(\Omega)$) and $\beta(\Omega)$ by using a geometrical number, and some properties of this number are discussed. Also connections between the corresponding contractive type mappings are given. We will show that $\delta(f) \leq \alpha(f)$ for any k-set contraction. In a special class of metric space, we prove that the set and the ball measures are equal.

Definition 4.1.1. For a complete metric space (M, ρ), define

$$\vartheta(M) := \sup\Big\{b > 0\text{: there exists } a > 1 \text{ such that for all } x, y \in M \text{ and all } r > 0,\ \rho(x,y) > r \text{ implies } \beta\big(\overline{\mathrm{B}}(x, br) \cap \overline{\mathrm{B}}(y, ar)\big) \leq 2r\Big\}.$$

$\vartheta(M)$ is a modification of the Lifschitz characteristic $\kappa(M)$. Obviously, $\vartheta(M) \geq \kappa(M) \geq 1$, and if M is a finite-dimensional Banach space, $\vartheta(M) = +\infty$. For a Banach space X, we have $\vartheta(X) \geq 1/(1-\delta_X(1))$ since $\kappa(X) \geq \kappa_0(X) \geq 1/(1-\delta_X(1))$.

We intend to relate $\alpha(\Omega)$ and $\beta(\Omega)$ by means of $\vartheta(M)$. The essential step is given by the next Lemma.

Lemma 4.1.2. *Let (M, ρ) be a complete metric space and let Ω be a bounded subset of M with diameter d. Then $\beta(\Omega) \leq 2d/\vartheta(M)$ (if $\vartheta(M) = +\infty$, this means $\beta(\Omega) = 0$).*

Proof. (Using the same idea as Lemma 2.1.1) If $\vartheta(M) \leq 1$ there is nothing to prove, so suppose that $\vartheta(M) > 1$. It suffices to prove $\beta(\Omega) \leq 2d/b$ for any $1 < b < \vartheta(M)$. By the definition of $\vartheta(M)$, there is $a > 1$ such that for all $x, y \in M$ and all $r > 0$, $\rho(x, y) > r$ implies that $\beta\big(\overline{\mathrm{B}}(y, ar) \cap \overline{\mathrm{B}}(x, br)\big) \leq 2r$.

If $a \geq b$. let $r = d/b < d$. Then there are points $x, y \in \Omega$ with $\rho(x, y) > r$. Hence, $\beta(\overline{B}(y, ad/b) \cap \overline{B}(x, d)) \leq 2d/b$. Since $a/b \geq 1$, $\Omega \subseteq \overline{B}(y, ad/b) \cap \overline{B}(x, d)$. Therefore $\beta(\Omega) \leq 2d/b$.

If $a < b$. Since $a > 1$, there is an integer $N \geq 1$ such that $a^N < b \leq a^{N+1}$. Let $\gamma \leq 1$ be such that $\gamma a^{N+1} = b$; note that $\gamma a > 1$. For $n = 1, 2, \ldots, N+1$, let $r_n = d/(\gamma a^n)$. We claim that $\beta(\Omega) \leq 2r_n$ for all n: $1 \leq n \leq N+1$. Indeed, for $n = 1$, $r_1 < d$, so there are $x, y \in \Omega$ with $\rho(x, y) > r_1$, thus $\beta(\overline{B}(y, ar_1) \cap \overline{B}(x, br_1)) \leq 2r_1$. Since $br_1 > ar_1 \geq d$, $\Omega \subseteq \overline{B}(y, ar_1) \cap \overline{B}(x, br_1)$, this proves the case $n = 1$. Now suppose the above has been shown for $n = i \leq N$, that is, $\beta(\Omega) \leq 2r_i$. Then for any $\varepsilon > 0$, there are $z_1, \ldots, z_m$ in M such that $\Omega \subseteq \bigcup_{j=1}^{m} B(z_j, r_i + \varepsilon)$. Therefore we have

$$\Omega = \bigcup_{j=1}^{m} (\Omega \cap B(z_j, r_i + \varepsilon)).$$

Let $\Omega_j = \Omega \cap B(z_j, r_i + \varepsilon)$. For each Ω_j $(1 \leq j \leq m)$, if $\rho(x, z_j) \leq r_{i+1} + \varepsilon$ for every $x \in \Omega_j$, then $\beta(\Omega_j) \leq 2(r_{i+1} + \varepsilon)$; if there is $x \in \Omega_j$ such that $\rho(x, z_j) > r_{i+1} + \varepsilon$, then

$$\beta(\overline{B}(z_j, a(r_{i+1} + \varepsilon)) \cap \overline{B}(x, b(r_{i+1} + \varepsilon))) \leq 2(r_{i+1} + \varepsilon).$$

For $y \in \Omega_j$, $\rho(y, z_j) \leq r_i + \varepsilon = ar_{i+1} + \varepsilon \leq a(r_{i+1} + \varepsilon)$ and $\rho(y, x) \leq d = \gamma a^{i+1} r_{i+1} \leq br_{i+1}$, since $\gamma a^{i+1} \leq \gamma a^{N+1} = b$. Hence $\Omega_j \subseteq \overline{B}(z_j, a(r_{i+1} + \varepsilon)) \cap \overline{B}(x, b(r_{i+1} + \varepsilon))$, therefore $\beta(\Omega_j) \leq 2(r_{i+1} + \varepsilon)$. We obtain that $\beta(\Omega) = \max_{1 \leq j \leq m} \beta(\Omega_j) \leq 2(r_{i+1} + \varepsilon)$, so $\beta(\Omega) \leq 2r_{i+1}$ by the arbitrariness of ε. Hence in finitely many steps, we obtain that $\beta(\Omega) \leq 2r_n$ for all n: $1 \leq n \leq N+1$. Since $r_{N+1} = d/b$, we have $\beta(\Omega) \leq 2d/b$.

Now we can connect $\alpha(\Omega)$ and $\beta(\Omega)$ by using $\vartheta(M)$.

Theorem 4.1.3. *Let (M, ρ) be a complete metric space. For every bounded subset Ω of M, we have $\beta(\Omega) \leq 2\alpha(\Omega)/\vartheta(M)$.*

Proof. For any $\delta > 0$, there are $\Omega_i \subseteq \Omega$, $i = 1, 2, \ldots, n$, such that $\Omega = \bigcup_{i=1}^{n} \Omega_i$ and

$\operatorname{diam}(\Omega_i) \le \alpha(\Omega)+\delta$. Let $d_i = \operatorname{diam}(\Omega_i)$ $(1 \le i \le n)$. By Lemma 4.1.2, we have $\beta(\Omega_i) \le 2d_i/\vartheta(M) \le 2(\alpha(\Omega)+\delta)/\vartheta(M)$. Hence $\beta(\Omega) = \max_{1 \le i \le n} \beta(\Omega_i) \le 2(\alpha(\Omega)+\delta)/\vartheta(M)$. The result follows since δ is arbitrary.

Remark. If M is not a locally compact metric space, then $\vartheta(M) \le 2$. In fact, there is a bounded set Ω in M which is not precompact, that is $\beta(\Omega) > 0$. Then $\vartheta(M) \le 2\alpha(\Omega)/\beta(\Omega) \le 2$.

If X is a Banach space, Danes [Dan-1] showed that $\beta(\Omega) \le 2(1-\delta_X(1))\alpha(\Omega)$. Since $\vartheta(X) \ge 1/(1-\delta_X(1))$, the above result improves on that of Danes. If H is an infinite-dimensional Hilbert space, $\vartheta(H) \ge \kappa(H) = \sqrt{2}$ (Theorem 2.1.8). Hence for any bounded subset Ω of H, we have $\beta(\Omega) \le \sqrt{2}\,\alpha(\Omega)$, this is a result in [Dan-2]. Later we will see that this is the best possible inequality in Hilbert space.

The next result is a direct consequence of Theorem 4.1.3.

Corollary 4.1.4. *Let M and E be two complete metric spaces. Let $f: M \longrightarrow E$ be a k-ball contraction for some $k>0$, then $\alpha(f) \le 2\beta(f)/\vartheta(M)$ and $\beta(f) \le 2\alpha(f)/\vartheta(E)$.*

Proof. For any bounded set Ω in M, by Theorem 4.1.3, we have

$$\alpha(f(\Omega)) \le \beta(f(\Omega)) \le \beta(f)\beta(\Omega) \le \beta(f)\, 2\alpha(\Omega)/\vartheta(M) \quad \text{and}$$

$$\beta(f(\Omega)) \le 2\alpha(f(\Omega))/\vartheta(E) \le 2\alpha(f)\alpha(\Omega)/\vartheta(E) \le 2\alpha(f)\beta(\Omega)/\vartheta(E).$$

If M is separable, α can be replaced by δ in Theorem 4.1.3.

Theorem 4.1.5. *Let (M,ρ) be a complete separable metric space. For every bounded set Ω in M, we have $\beta(\Omega) \le 2\delta(\Omega)/\vartheta(M)$.*

Proof. Without loss generality we suppose Ω is an infinite set. By Proposition 1.3.8, there exists a β-minimal subset Ω' of Ω such that $\beta(\Omega') = \beta(\Omega)$. From Proposition 1.3.7, there is an α-minimal subset $\Omega'' \subseteq \Omega'$. Thus, by Theorem 4.1.3

and Lemma 3.2.3, we have

$$\beta(\Omega)=\beta(\Omega')=\beta(\Omega'')\leq 2\alpha(\Omega'')/\vartheta(M)=2\delta(\Omega'')/\vartheta(M)\leq 2\delta(\Omega)/\vartheta(M).$$

Next we show a relation between k-set contractions and k-δ-contractions.

Theorem 4.1.6. *Every k-set contraction f from a complete metric space M to another complete metric space E is a k-δ-contraction, that is, $\delta(f)\leq\alpha(f)$.*

Proof. Let $\Omega\subseteq M$, if $\delta(f(\Omega))=0$, then $\delta(f(\Omega))\leq k\delta(\Omega)$. Now suppose $\delta(f(\Omega))\neq 0$. For any $\varepsilon>0$, there is $\Omega_\varepsilon\subseteq\Omega$ such that $f(\Omega_\varepsilon)$ is α-minimal and $\delta(f(\Omega))\leq\alpha(f(\Omega_\varepsilon))+\varepsilon$ (Lemma 3.2.3). Let $\{x_n\}$ be a sequence in Ω_ε satisfying: $f(x_n)\neq f(x_m)$ whenever $n\neq m$. We can suppose that $\{x_n\}$ is α-minimal, otherwise using Proposition 1.3.7, take a subsequence. Then using Lemma 3.2.3 again, we obtain

$$\delta(f(\Omega))\leq\alpha(\{f(x_n)\})+\varepsilon\leq k\alpha(\{x_n\})+\varepsilon\leq k\delta(\Omega)+\varepsilon.$$

Let $\varepsilon\longrightarrow 0$, we have $\delta(f(\Omega))\leq k\delta(\Omega)$. Hence f is a k-δ-contraction.

From Corollary 4.1.4 and Theorem 4.1.6, we also have $\delta(f)\leq 2\beta(f)/\vartheta(M)$ for any k-ball contraction f.

Next we give the value of $\vartheta(M)$ when M is a Hilbert space.

Theorem 4.1.7. *Let H be an infinite-dimensional Hilbert space, then $\vartheta(H)=\sqrt{2}$.*

Proof. $\vartheta(H)\geq\kappa(H)=\sqrt{2}$ (Theorem 2.1.8). Next we prove $\vartheta(H)\leq\sqrt{2}$. There is a sequence $\{e_n\}\subseteq H$ such that $(e_n, e_m)=0$ if $n\neq m$, $\|e_n\|=1$ and for any $x\in H$, $\sum_{n=1}^{\infty}|(x, e_n)|^2\leq\|x\|^2$. Hence we have $(x, e_n)\longrightarrow 0$ $(n\longrightarrow\infty)$ for any $x\in H$. We claim that $\alpha(\{e_n\})=\sqrt{2}$ and $\beta(\{e_n\})=2$.

In fact, for any subsequence $\{e_{n_k}\}$ of $\{e_n\}$, $\mathrm{diam}(\{e_{n_k}\})=\sqrt{2}$, since

$\|e_n-e_m\|^2=2$ whenever $n\neq m$. So $\alpha(\{e_n\})=\sqrt{2}$. $\beta(\{e_n\})\le 2$ since $\|e_n\|=1$. If $\beta(\{e_n\}):=2a<2$. Let $\varepsilon>0$ be so that $a+\varepsilon<1$, there are $x_i\in H$, $i=1,\dots,p$, $p\in\mathbb{N}$, such that $\{e_n\}\subseteq \bigcup_{i=1}^{p} B(x_i, a+\varepsilon)$. There is at least one ball $B(x_i, a+\varepsilon)$ $(1\le i\le p)$ containing a subsequence $\{e_{n_k}\}$ of $\{e_n\}$. For this x_i, we have

$$1>(a+\varepsilon)^2\ge \|e_{n_k}-x_i\|^2=1+\|x_i\|^2-2(x_i, e_{n_k})\longrightarrow 1+\|x_i\|^2 \quad (k\longrightarrow\infty).$$

This is a contradiction, so $\beta(\{e_n\})=2$.

By Theorem 4.1.3, we have $\vartheta(H)\le 2\alpha(\{e_n\})/\beta(\{e_n\})=\sqrt{2}$.

Remark. The above proof also shows that $\beta(\Omega)\le\sqrt{2}\,\alpha(\Omega)$ is best possible in infinite-dimensional Hilbert spaces.

If X is an infinite-dimensional Banach space, we can define a number $\vartheta_0(X)$ corresponding to $\kappa_0(X)$ as following:

$$\vartheta_0(X)=\inf\left\{\vartheta(C):\ C \text{ a closed, bounded, convex and noncompact subset of } X\right\}.$$

Since $\vartheta(C)\ge\kappa(C)$ for any C, $\vartheta_0(X)\ge\kappa_0(X)$. Next we show the relation between $\vartheta_0(X)$ and $N_\beta(X)$.

Proposition 4.1.8. *Let X be an infinite-dimensional Banach space, then $\vartheta_0(X)\le N_\beta(X)$. Hence if $\vartheta_0(X)>1$, X has normal structure.*

Proof. Let C be any closed, bounded, convex and noncompact subset of X. From Lemma 4.1.2, we have $\beta_C(C)\le 2\mathrm{diam}(C)/\vartheta(C)$. By the definitions of $\vartheta_0(X)$ and $N_\beta(X)$, we obtain $\vartheta_0(X)\le N_\beta(X)$.

Proposition 4.1.9. *For any infinite-dimensional Banach space X, let*

$$\xi(X)=\sup\Big\{b>0: \textit{there exists } a>1 \textit{ such that for all } y\in X \textit{ with } \|y\|>1,\ \beta\big(\overline{B}(0,b)\cap\overline{B}(y,a)\big)\le 2\Big\}$$

and let

$$\zeta(X)=\sup\Big\{b>0: \textit{there exists } a>1 \textit{ such that for all } y\in X \textit{ with } \|y\|>1,\ \overline{B}(0,b)\cap\overline{B}(y,a)\subseteq \bigcup_{t\in[0,1]} \overline{B}(ty,1)\Big\}.$$

Then $\vartheta_0(X)\le\xi(X)=\vartheta(X)$ *and* $\vartheta_0(X)\ge\zeta(X)$.

Proof. First we prove $\xi(X)=\vartheta(X)$. Obviously $\xi(X)\ge\vartheta(X)$. For any $b: 0<b<\xi(X)$, there is $a>1$ such that for all $y\in X$ with $\|y\|>1$, $\beta(\overline{B}(0,b)\cap\overline{B}(y,a))\le 2$. For any $u, v\in X$ and any $r>0$ with $\|u-v\|>r$, we have

$$\begin{aligned}\overline{B}(u,br)\cap\overline{B}(v,ar)&=u+\overline{B}(0,br)\cap\overline{B}(v-u,ar)\\ &=u+r\big(\overline{B}(0,b)\cap\overline{B}((v-u)/r,a)\big).\end{aligned}$$

Thus $\beta\big(\overline{B}(u,br)\cap\overline{B}(v,ar)\big)=r\beta\big(\overline{B}(0,b)\cap\overline{B}((v-u)/r,a)\big)\le 2r$. Then $\vartheta(X)\ge b$, and $\vartheta(X)\ge\xi(X)$ by the arbitrariness of b.

Now we prove $\vartheta_0(X)\le\xi(X)$. We only need to prove $\vartheta(\overline{B}(0,4))\le\xi(X)$. For any $b: 0<b<\vartheta(\overline{B}(0,4))$, there is $a>1$ such that for any $x, y\in\overline{B}(0,4)$ and any $r>0$, $\|x-y\|>r$ implies $\beta(\overline{B}(x,br)\cap\overline{B}(y,ar))\le 2r$. For a and b as above, if $1<\|y\|<4$, then $\beta(\overline{B}(0,b)\cap\overline{B}(y,a))\le 2$; if $\|y\|\ge 4$, then $\overline{B}(0,b)\cap\overline{B}(y,3/2)=\emptyset$ since $b<2$. Hence $\vartheta(\overline{B}(0,4))\le\xi(X)$.

Lastly we prove $\vartheta_0(X)\ge\zeta(X)$. For any b: $0<b<\zeta(X)$, there is $a>1$ such that for all $y\in X$ with $\|y\|>1$, $\overline{B}(0,b)\cap\overline{B}(y,a)\subseteq\bigcup_{t\in[0,1]}\overline{B}(ty,1)$. For any $\varepsilon>0$, there are $t_1,\dots,t_n\in[0,1]$ such that if $t\in[0,1]$, then for some t_i $(1\le i\le n)$, $|t-t_i|<\varepsilon/\|y\|$. Therefore, if $\|x-ty\|\le 1$, $\|x-t_iy\|\le\|x-ty\|+|t-t_i|\,\|y\|<1+\varepsilon$. Then we have

$$\overline{B}(0,b)\cap\overline{B}(y,a)\subseteq\bigcup_{i=1}^{n}B(t_iy,1+\varepsilon).$$

Let C be any closed, bounded, convex and noncompact subset of X. For

a and b as above, and any $u, v \in C$ and $r>0$, if $\|u-v\|>r$, we have $\|u-v\|/r>1$. Hence

$$\begin{aligned}\overline{\mathrm{B}}(u, br)\cap\overline{\mathrm{B}}(v, ar) &= u + r\left(\overline{\mathrm{B}}(0, b)\cap\overline{\mathrm{B}}((u-v)/r, a)\right)\\ &\subseteq u + r\left(\bigcup_{i=1}^{n} \mathrm{B}(t_i(u-v)/r, 1+\varepsilon)\right)\\ &\subseteq \bigcup_{i=1}^{n} \mathrm{B}(u+t_i(u-v), r(1+\varepsilon)).\end{aligned}$$

Since C is convex, $u+t_i(u-v)\in C$. Hence $\beta_C\left(\overline{\mathrm{B}}(u, br)\cap\overline{\mathrm{B}}(v, ar)\right)\leq 2r(1+\varepsilon)$. We obtain $\beta_C\left(\overline{\mathrm{B}}(u, br)\cap\overline{\mathrm{B}}(v, ar)\right)\leq 2r$ as ε is arbitrary. Therefore $\vartheta(C)\geq\zeta(X)$. As C is arbitrary, we obtain $\vartheta_0(X)\geq\zeta(X)$.

The above proposition may be used to estimate the values of $\vartheta(X)$ and $\vartheta_0(X)$ for a Banach space X. We tried to use it obtain an example where $\vartheta(X)>\kappa(X)$ in an infinite-dimensional Banach space, but did not succeed.

4.2. Connections between several measures of noncompactness in Banach spaces

Via the results in sections 3.2 and 4.1, we see that $\beta(\Omega)\leq 2\alpha(\Omega)$ can be improved to $\beta(\Omega)\leq 2\alpha(\Omega)/\vartheta(M)$ in a metric space M and $\beta(\Omega)\leq 2\alpha(\Omega)/N_\beta(X)$ in a Banach space X; also α can be replaced by δ. We will obtain other improved inequalities of this kind in Banach space by using some other geometrical coefficients of the underlying spaces. Also some relationships between these geometrical numbers are given.

In this section, we consider infinite-dimensional Banach spaces.

First we give a connection between $\alpha(\Omega)$ and $\beta(\Omega)$ by using $W(X)$, which is taken from [WebZ-2].

Theorem 4.2.1. *Let X be an infinite-dimensional, separable, reflexive, Banach space. Then for every bounded subset Ω of X, $\beta(\Omega)\le 2\alpha(\Omega)/W(X)$.*

Proof. Let $\eta(X)=\sup\{\beta(\Omega)/\alpha(\Omega)\colon \Omega\subseteq X$ bounded and noncompact$\}$. It suffices to prove $2/\eta(X)\ge W(X)$. For any ε: $0<\varepsilon<\eta(X)$, there is a bounded subset Ω of X such that $\beta(\Omega)\ge(\eta(X)-\varepsilon)\alpha(\Omega)>0$. We can suppose that $\beta(\Omega)\ge 1$, otherwise, consider $\Omega':=\Omega/\beta(\Omega)$, $\beta(\Omega')=1$ and $\alpha(\Omega')=\alpha(\Omega)/\beta(\Omega)$. As X is separable, there exists $\Omega_1\subseteq\Omega$ with Ω_1 β-minimal and $\beta(\Omega_1)=\beta(\Omega)$; (Proposition 1.3.8). There is $\Omega_2\subseteq\Omega_1$ with Ω_2 α-minimal and $0<\alpha(\Omega_2)\le\alpha(\Omega_1)$ (Proposition 1.3.7). As Ω_2 is α-minimal and β-minimal, and, since X is separable and reflexive, there is an infinite sequence $\{x_n\}$ of distinct points of Ω_2 satisfies that

1) $\alpha(\Omega_2)\ge\operatorname{diam}\{x_n\}-\varepsilon$ (Lemma 1.3.9);

2) $\{x_n\}$ is weakly convergent; and

3) for any $z\in X$, $\lim_{n\to\infty}\|x_n-z\|$ exists (Lemma 3.3.2).

Then

$$\beta(\Omega)=\beta(\{x_n\})\le 2\inf_{z\in X}\lim_{n\to\infty}\|x_n-z\|\le 2\operatorname{diam}\{x_n\}/W(X)$$
$$\le 2(\alpha(\Omega_2)+\varepsilon)/W(X)\le 2(\alpha(\Omega)+\varepsilon)/W(X)$$

Thus $2/(\eta(X)-\varepsilon)\ge 2\alpha(\Omega)/\beta(\Omega)\ge W(X)-2\varepsilon/\beta(\Omega)\ge W(X)-2\varepsilon$. As ε is arbitrary, this shows $2/\eta(X)\ge W(X)$.

Since we know that $N_\beta(X)=WCS(X)\le W(X)$, the above result improves on $\beta(\Omega)\le 2\alpha(\Omega)/N_\beta(X)$, but it only considers separable spaces.

Corollary 4.2.2. *Let X be an infinite-dimensional, separable, reflexive, Banach space. For every bounded subset Ω of X, $\beta(\Omega)\le 2\delta(\Omega)/W(X)$.*

Proof. For any bounded subset Ω in X, by Propositions 1.3.7 and 1.3.8, there is an α and β-minimal subset $\Omega' \subseteq \Omega$ so that $\beta(\Omega)=\beta(\Omega')$. Thus by Theorem 4.2.1 and Lemma 3.2.3, we have $\beta(\Omega)=\beta(\Omega')\leq 2\alpha(\Omega')/W(X)\leq 2\delta(\Omega)/W(X)$.

Next we connect $\alpha(\Omega)$ and $\beta(\Omega)$ by $\overline{N}(X)$. We know that $N(X)\leq\overline{N}(X)$ and $N(X)\leq N_\beta(X)$, but we do not know whether $\overline{N}(X)$ and $N_\beta(X)$ are comparable.

Theorem 4.2.3. *Let X be a Banach space, then for any bounded subset Ω of X, we have $\beta(\Omega)\leq 2\alpha(\Omega)/\overline{N}(X)$.*

Proof. For any $\varepsilon>0$, there are $\Omega_i\subseteq\Omega$, $i=1,2,\dots,n$, such that $\Omega=\bigcup_{i=1}^{n}\Omega_i$ and $\mathrm{diam}(\Omega_i)<\alpha(\Omega)+\varepsilon$. There is $j: 1\leq j\leq n$, such that $\beta(\Omega_j)=\max_{1\leq i\leq n}\beta(\Omega_i)$. Then

$$\beta(\Omega)=\beta(\Omega_j)=\beta(\overline{\mathrm{co}}\Omega_j)\leq 2r(\overline{\mathrm{co}}\Omega_j, X)\leq 2\mathrm{diam}(\overline{\mathrm{co}}\Omega_j)/\overline{N}(X)$$
$$=2\mathrm{diam}(\Omega_j)/\overline{N}(X)<2(\alpha(\Omega)+\varepsilon)/\overline{N}(X).$$

Since ε is arbitrary, we obtain $\beta(\Omega)\leq 2\alpha(\Omega)/\overline{N}(X)$.

Now we define a geometrical number in a Banach space X, and will use it to relate several measures of noncompactness.

Definition 4.2.4. Let X be a Banach space, define

$$K_\beta(X):=\inf\Big\{2\mathrm{diam}(C)/\beta(C):\ C \text{ a closed, bounded, convex and noncompact subset of } X\Big\}.$$

Obviously $1\leq N_\beta(X)\leq K_\beta(X)\leq 2$, since $\beta(C)\leq\beta_C(C)$ and $\beta(\mathrm{B}_{\mathrm{X}})=\mathrm{diam}(\mathrm{B}_{\mathrm{X}})=2$. Next we will see that $K_\beta(X)$ can be 1 or 2, also it is possible to have $N_\beta(X)<K_\beta(X)$

Example 4.2.5. $K_\beta(c_0)=1$. In fact, consider the sequence $\{e_n\}$ in c_0, where

$e_n=(\delta_{nm})$ with $\delta_{nm}=0$ if $n\neq m$ and $\delta_{nm}=1$ if $n=m$. Obviously $\mathrm{diam}(\{e_n\})=1$. Also in Chapter one we showed that $\beta(\{e_n\})=2$. Let $C=\overline{\mathrm{co}}\{x_n\}$, then

$$K_\beta(c_0)\leq 2\mathrm{diam}(C)/\beta(C)=2\mathrm{diam}(\{e_n\})/\beta(\{e_n\})=1$$

Theorem 4.2.6. $K_\beta(l^1)=2$.

Proof. We only need to prove that $K_\beta(l^1)\geq 2$. Let Ω be any bounded set in l^1, it suffices to prove $\beta(\Omega)\leq\mathrm{diam}(\Omega)$. We use a method similar to that of [Ben-2] to prove this. We can suppose $\beta(\Omega)>0$. Since l^1 is separable, by Proposition 1.3.8, there is a β-minimal subset Ω' of Ω such that $\beta(\Omega')=\beta(\Omega)$. Using Proposition 1.3.7, there exists an α-minimal subset $\Omega''\subseteq\Omega'$. Let $\{x_n\}$ be a sequence of distinct points of Ω''. Using a diagonal method, we can find a subsequence $\{y_n\}$ (say) of $\{x_n\}$ such that $y_{nk}\longrightarrow v_k$ $(n\longrightarrow\infty)$ for each $k\in\mathbb{N}$, where $y_n=(y_{nk})$. Let $v=(v_k)$, then $v\in l^1$. In fact, for any fixed $m\in\mathbb{N}$, there is $n\in\mathbb{N}$ so that $|y_{nk}-v_k|<1/m$ for $k=1, 2, \ldots, m$. Hence we have

$$\sum_{k=1}^{m}|v_k|\leq\sum_{k=1}^{m}|v_k-y_{nk}|+\sum_{k=1}^{m}|y_{nk}|\leq 1+\|y_n\|\leq 1+M,$$

where $M=\sup_{n\geq 1}\|y_n\|<+\infty$.

By Lemma 3.3.2, we can suppose $\phi(z):=\lim_{n\to\infty}\|y_n-z\|$ exists for any $z\in l^1$ (otherwise take a subsequence). We claim $\phi(v)\leq\phi(z)$ for any $z\in l^1$. Let $z=(z_k)$. For any $\varepsilon>0$, there is $m\in\mathbb{N}$ so that

$$\sum_{k=m+1}^{\infty}|v_k|<\varepsilon \quad\text{and}\quad \sum_{k=m+1}^{\infty}|z_k|<\varepsilon.$$

There is $n\in\mathbb{N}$ such that $\|y_n-z\|<\phi(z)+\varepsilon$, $\|y_n-v\|>\phi(v)-\varepsilon$ and $|y_{nk}-v_k|<\varepsilon/m$ for $k=1, 2, \ldots, m$. Hence we have

$$\phi(v) < \|y_n - v\| + \varepsilon = \sum_{k=1}^{m} |y_{nk} - v_k| + \sum_{k=m+1}^{\infty} |y_{nk} - v_k| + \varepsilon$$

$$< \varepsilon + \sum_{k=m+1}^{\infty} |y_{nk} - z_k| + \sum_{k=m+1}^{\infty} |v_k| + \sum_{k=m+1}^{\infty} |z_k| + \varepsilon$$

$$< 4\varepsilon + \|y_n - z\| < \phi(z) + 5\varepsilon.$$

Obviously, $\beta(\{y_n\}) = 2\phi(v)$, so $\beta(\Omega) = 2\phi(v)$. We finish the proof by showing that $\mathrm{diam}(\Omega) \geq 2\phi(v)$. For any $\varepsilon > 0$, there is $m \in \mathbb{N}$ so that $\|y_n - v\| > \phi(v) - \varepsilon$ whenever $n \geq m$. For this fixed m, there is $p \in \mathbb{N}$ such that $\sum_{k=p+1}^{\infty} |y_{mk} - v_k| < \varepsilon$. For this p, there is $n > m$ such that $|y_{nk} - v_k| < \varepsilon/p$ for $k = 1, 2, \ldots, p$. Thus we have

$$\mathrm{diam}(C) \geq \|y_n - y_m\| = \sum_{k=1}^{p} |y_{nk} - y_{mk}| + \sum_{k=p+1}^{\infty} |y_{nk} - y_{mk}|$$

$$\geq \left(\sum_{k=1}^{p} |v_k - y_{mk}| - \sum_{k=1}^{p} |y_{nk} - v_k| \right) + \left(\sum_{k=p+1}^{\infty} |y_{nk} - v_k| - \sum_{k=p+1}^{\infty} |v_k - y_{mk}| \right)$$

$$\geq \left(\sum_{k=1}^{\infty} |v_k - y_{mk}| - \sum_{k=p+1}^{\infty} |v_k - y_{mk}| - \varepsilon \right) + \left(\sum_{k=1}^{\infty} |y_{nk} - v_k| - \sum_{k=1}^{p} |y_{nk} - v_k| - \varepsilon \right)$$

$$\geq \|y_m - v\| + \|y_n - v\| - 4\varepsilon \geq 2\phi(v) - 6\varepsilon.$$

We obtain $\mathrm{diam}(\Omega) \geq 2\phi(v)$ since ε is arbitrary.

$N_\beta(l^1) = 1$ since l^1 is not reflexive, so $N_\beta(l^1) < K_\beta(l^1)$. We will see other examples where $N_\beta(X) < K_\beta(X)$ in section 4.4.

Now we use $K_\beta(X)$ to connect $\alpha(\Omega)$ and $\beta(\Omega)$.

Theorem 4.2.7. *Let X be a Banach space. Then for any bounded subset Ω of X, we have $\beta(\Omega) \leq 2\alpha(\Omega)/K_\beta(X)$. Also $K_\beta(X)$ is the best possible constant to satisfy the inequality in the sense that, if a number $\mu(X)$ is such that $\beta(\Omega) \leq 2\alpha(\Omega)/\mu(X)$ for all bounded subset of X, then $\mu(X) \leq K_\beta(X)$, that is*

$$H(X):=\inf\left\{2\alpha(\Omega)/\beta(\Omega):\ \Omega \text{ a bounded subset of } X \text{ with } \beta(\Omega)\neq 0\right\}=K_\beta(X).$$

<u>Proof</u>. Assume $\beta(\Omega)>0$. For any $\varepsilon>0$, there are $\Omega_1, \ldots, \Omega_n \subseteq \Omega$ so that $\Omega=\bigcup_{i=1}^{n}\Omega_i$ and $\mathrm{diam}(\Omega_i)\leq\alpha(\Omega)+\varepsilon$ for all i, $1\leq i\leq n$. Thus there is a j: $1\leq j\leq n$ such that

$$\begin{aligned}\beta(\Omega)&=\beta(\Omega_j)=\beta(\overline{\mathrm{co}}\,\Omega_j)\leq 2\mathrm{diam}(\overline{\mathrm{co}}\,\Omega_j)/K_\beta(X)\\&=2\mathrm{diam}(\Omega_j)/K_\beta(X)\leq 2(\alpha(\Omega)+\varepsilon)/K_\beta(X).\end{aligned}$$

Hence $\beta(\Omega)\leq 2\alpha(\Omega)/K_\beta(X)$ by the arbitrariness of ε. This proves that $K_\beta(X)\leq 2\alpha(\Omega)/\beta(\Omega)$ for any bounded subset Ω of X with $\beta(\Omega)\neq 0$, and so $K_\beta(X)\leq H(X)$. $H(X)\leq K_\beta(X)$ is obvious.

<u>Remark</u>. In [BenA], Benavides and Ayerbe considered the following geometrical number $\lambda(M)$ for a metric space M:

$$\lambda(M)=\sup\left\{\beta(A)/\alpha(A) : A \text{ a bounded, } \alpha\text{-minimal and noncompact subset of } M\right\}.$$

They proved that $\beta(\Omega)\leq\lambda(M)\alpha(\Omega)$ hold in a separable metric space M. If X is a separable Banach space, then $\lambda(X)=2/K_\beta(X)$. In fact, $2/\lambda(X)\leq K_\beta(X)$ from Theorem 4.2.7. But we have

$$2/\lambda(X)=\inf\left\{2\alpha(A)/\beta(A) : A \text{ a bounded, } \alpha\text{-minimal and noncompact subset of } X\right\},$$

so $2/\lambda(X)\geq H(X)=K_\beta(X)$.

We intend to replace α by δ in Theorem 4.2.7, the following analogue to Lemma 3.2.4 is needed.

<u>Lemma 4.2.8</u>. *Let X be an infinite-dimensional Banach space, Ω a bounded subset of X with $\beta(\Omega)=2a>0$. Then for any r: $0<r<a$, there is a sequence $\{x_n\}\subseteq\Omega$ such that $\beta_Y(\overline{\mathrm{co}}\{x_n\})=\beta_Y(\{x_n\})\geq 2r$, where $Y=\overline{\mathrm{span}}\{x_n\}$.*

Proof. We construct a sequence $\{x_n\}$ satisfying the conclusion of the lemma. Let x_1 be any point in Ω. Suppose $x_1, \ldots, x_n$ have been obtained and let $Y_n = \text{span}\{x_1, \ldots, x_n\}$. We claim that there exists $x_{n+1} \in \Omega$ such that $d(x_{n+1}, Y_n) > r$. In fact, otherwise, $d(x, Y_n) \leq r$ for all $x \in \Omega$. Then for every x in Ω, there is $x' \in Y_n$ such that $\|x - x'\| = d(x, Y_n) \leq r$. Let $\Omega' = \{x' \in Y_n$: there exists $x \in \Omega$, $\|x - x'\| \leq r\}$. Ω' is bounded so Ω' is precompact, that is $\beta_{Y_n}(\Omega') = 0$. For $\varepsilon = (a-r)/2 > 0$, there are $x'_1, \ldots, x'_m$ in Y_n such that $\Omega' \subseteq \bigcup_{i=1}^{m} B(x'_i, \varepsilon)$. For any $x \in \Omega$, there is a $x' \in \Omega'$ such that $\|x - x'\| \leq r$, and for this x', there is x'_i such that $\|x' - x'_i\| \leq \varepsilon$, so $\|x - x'_i\| \leq \|x - x'\| + \|x' - x'_i\| \leq r + \varepsilon = (a+r)/2$, that is

$$\Omega \subseteq \bigcup_{i=1}^{m} B(x'_i, (a+r)/2).$$

Hence $\beta(\Omega) \leq a + r < 2a$, a contradiction.

By induction, we obtain a sequence $\{x_n\} \subseteq \Omega$ satisfying $d(x_{n+1}, Y_n) > r$ for all $n = 1, 2, \ldots$, where $Y_n = \text{span}\{x_1, \ldots, x_n\}$. We prove $\beta_Y(\{x_n\}) \geq 2r$, where $Y = \overline{\text{span}}\{x_n\}$. If $\beta_Y(\{x_n\}) = 2b < 2r$, let $\delta > 0$ be such that $b + 2\delta < r$. Then there are $y_j \in Y$, $j = 1, \ldots, l$, such that $\{x_n\} \subseteq \bigcup_{j=1}^{l} B(y_j, b+\delta)$. For every $y_j \in Y$, there is a $w_j \in Y_{p_j}$ such that $\|y_j - w_j\| \leq \delta$, hence $\{x_n\} \subseteq \bigcup_{j=1}^{l} B(w_j, b+2\delta)$. Let $N = \max\{p_j : 1 \leq j \leq l\}$, then for all $j = 1, \ldots, l$, $w_j \in Y_N$. So $\|x_{N+1} - w_j\| \geq d(x_{N+1}, Y_N) > r > b + 2\delta$ for all j, $1 \leq j \leq l$, that is, $x_{N+1} \notin \bigcup_{j=1}^{l} B(w_j, b+2\delta)$. This is a contradiction.

Definition 4.2.9. Let X be a Banach space, define

$$K^0_\beta(X) := \inf\Big\{K_\beta(Y)\text{: where } Y \text{ is an infinite-dimensional, separable and closed subspace of } X\Big\}.$$

Theorem 4.2.10. *Let X be a Banach space. then for any bounded subset Ω in X, we have $\beta(\Omega)\leq 2\delta(\Omega)/K_{\beta}^{0}(X)$. If X is separable, we have $\beta(\Omega)\leq 2\delta(\Omega)/K_{\beta}(X)$.*

Proof. Without loss of generality, we suppose that $\beta(\Omega):=2a>0$. By Lemma 4.2.8, for any ε: $0<\varepsilon<a$, there is a sequence $\{x_n\}\subseteq\Omega$ such that $\beta_{Y}(\{x_n\})\geq 2(a-\varepsilon)$, where $Y=\overline{\mathrm{span}}\{x_n\}$. From Propositions 1.3.7 and 1.3.8, there is an α and β-minimal subsequence $\{y_n\}\subseteq\{x_n\}$ so that $\beta_{Y}(\{y_n\})=\beta_{Y}(\{x_n\})$. Hence, by Theorem 4.2.7 and Lemma 3.2.3, we have

$$\begin{aligned}\beta(\Omega)&\leq\beta_{Y}(\{x_n\})+2\varepsilon=\beta_{Y}(\{y_n\})+2\varepsilon\\&\leq 2\alpha(\{y_n\})/K_{\beta}(Y)+2\varepsilon\leq 2\delta(\Omega)/K_{\beta}^{0}(X)+2\varepsilon.\end{aligned}$$

If X is separable, there is an α and β minimal subset Ω' of Ω such that $\beta(\Omega)=\beta(\Omega')$, so by Theorem 4.2.7 and Lemma 3.2.3, we have

$$\beta(\Omega)=\beta(\Omega')\leq 2\alpha(\Omega')/K_{\beta}(X)\leq 2\delta(\Omega)/K_{\beta}(X).$$

Now we give some connections between the geometrical numbers studied above to relate various measures of noncompactness.

Proposition 4.2.11. *Let X be a Banach space, then $K_{\beta}(X)\geq\vartheta(X)$, $K_{\beta}(X)\geq\overline{N}(X)$ and $K_{\beta}(X)\geq K_{\beta}^{0}(X)\geq N_{\beta}(X)$. If X is reflexive and separable, then $K_{\beta}(X)\geq W(X)$.*

Proof. By Theorems 4.1.3 and 4.2.7, we have $\vartheta(X)\leq K_{\beta}(X)$; by Theorems 4.2.3 and 4.2.7, we have $K_{\beta}(X)\geq\overline{N}(X)$. From Theorems 4.2.1 and 4.2.7, we have $W(X)\leq K_{\beta}(X)$ when X is a reflexive and separable Banach space.

Next we prove $K_{\beta}(X)\geq K_{\beta}^{0}(X)$. For any ε: $0<\varepsilon<1$, there is a closed, bounded, convex and noncompact subset C in X so that $K_{\beta}(X)>2\mathrm{diam}(C)/\beta(C)-\varepsilon$. Without loss of generality we can suppose that $\beta(C)=1$, otherwise, take $C'=C/\beta(C)$. By Lemma 4.2.8, there is a sequence $\{x_n\}\subseteq C$ such that $\beta_{Y}(D)\geq 1-\varepsilon$, where $Y=\overline{\mathrm{span}}\{x_n\}$ and $D=\overline{\mathrm{co}}\{x_n\}$. Thus, noting that $\beta_{Y}(D)/(\beta_{Y}(D)+\varepsilon)\geq 1-\varepsilon$, we have

$$K_\beta(X) > 2\mathrm{diam}(D)/(\beta_Y(D)+\varepsilon)-\varepsilon \geq K_\beta(Y)\big(\beta_Y(D)/(\beta_Y(D)+\varepsilon)\big)-\varepsilon \geq K_\beta^0(X)(1-\varepsilon)-\varepsilon.$$

The result follows by the arbitrariness of ε.

Lastly we show $K_\beta^0(X) \geq N_\beta(X)$. Let Y be any infinite-dimensional, separable and closed subspace of X. For any closed, bounded, convex and noncompact subset C of Y, we have $\beta_Y(C) \leq \beta_C(C)$. Hence $K_\beta(Y) \geq N_\beta(Y) \geq N_\beta(X)$. By the definition of $K_\beta^0(X)$, we obtain that $K_\beta^0(X) \geq N_\beta(X)$.

Remark. From above result and results in chapter one, we have

$$K_\beta(X) \geq K_\beta^0(X) \geq N_\beta(X) \geq N(X) \geq \kappa_0(X) \geq 1/(1-\delta_X(1)).$$

In certain special classes of Banach space, some of these geometrical numbers are equal. To show this, we next give the definition of an Opial space.

Definition 4.2.12. [O] A Banach space is said to satisfy *Opial's condition* or to be an *Opial space*, if for every sequence $\{x_n\}$ in X weakly convergent to x, it follows that for any y in X,

$$\liminf_{n\to\infty}\|x_n-y\| \geq \liminf_{n\to\infty}\|x_n-x\|.$$

A Banach space with a weakly continuous duality mapping is an Opial space [O], so this includes Hilbert and l^p spaces. Gossez and Lami Dozo [GosD] showed the converse is not true and studied Opial spaces further.

Theorem 4.2.13. *If X is an infinite-dimensional reflexive Opial space, then*

$$K_\beta(X) = K_\beta^0(X) = N_\beta(X) = WCS(X) = W(X).$$

Thus if $X=H$ is a Hilbert space, then $K_\beta(H)=K_\beta^0(H)=W(H)=\sqrt{2}$; and for $1<p<\infty$, $K_\beta(l^p)=K_\beta^0(l^p)=W(l^p)=2^{1/p}$.

Proof. By Theorems 3.3.4 and 4.2.11, we only need to prove $N_\beta(X)=K_\beta(X)$ and $WCS(X)=W(X)$. For any closed, bounded, convex and noncompact subset C of X, we intend to prove $\beta_C(C)=\beta(C)$. Then by the definitions of $N_\beta(X)$ and $K_\beta(X)$, it is immediate that the two number are equal. Let $\beta_C(C):=2a>0$. Then for any r: $0<r<a$, there is a sequence $\{x_n\}\subseteq C$ such that $\beta_D(D)\geq 2r$, where $D=\overline{\mathrm{co}}\{x_n\}$ (Lemma 3.2.4). By Proposition 1.3.8, there is a β-minimal subset A of D such that $\beta_D(D)=\beta_D(A)$. As X is reflexive, by taking subsequences several times, there is a sequence $\{z_n\}\subseteq A$ such that $z_n\neq z_m$ if $n\neq m$, $\{z_n\}$ converges weakly to $z\in D$, and $\lim_{n\to\infty}\|z_n-z\|$ exists. We claim that $\beta_D(\{z_n\})=\beta(\{z_n\})$. In fact, for any $b>\beta(\{z_n\})/2$, there are $w_1, \ldots, w_m\in X$ so that $\{z_n\}\subseteq \bigcup_{j=1}^{m} \mathrm{B}(w_j, b)$. There is some $\mathrm{B}(w_j, b)$ $(1\leq j\leq m)$ containing a subsequence $\{z_{n_k}\}$, then because of Opial's condition

$$\lim_{n\to\infty}\|z_{n_k}-z\|\leq \liminf_{n\to\infty}\|z_{n_k}-w_j\|\leq b.$$

Therefore, $\beta_D(\{z_n\})\leq 2\lim_{n\to\infty}\|z_n-z\|\leq 2b$, and $\beta_D(\{z_n\})\leq\beta(\{z_n\})$ since b is arbitrary. We obtain the claimed result as $\beta_D(\{z_n\})\geq\beta(\{z_n\})$ is obvious. Hence

$$2r\leq\beta_D(D)=\beta_D(A)=\beta_D(\{z_n\})=\beta(\{z_n\})\leq\beta(C).$$

Let $r\longrightarrow a$, we obtain $\beta_C(C)\leq\beta(C)$. Since $\beta_C(C)\geq\beta(C)$ is obvious, we have $\beta_C(C)=\beta(C)$.

Now we prove $WCS(X)=W(X)$. Since $WCS(X)\leq W(X)$, we only need to show that $WCS(X)\geq W(X)$. By Lemma 3.3.3, we have

$$WCS(X)=\inf\left\{\mathrm{diam}\{x_n\} : \|x_n\|=1 \text{ and } w\text{-}\lim_{n\to\infty} x_n=0\right\}.$$

So for any $\varepsilon>0$, there is a sequence $\{x_n\}$ with $\|x_n\|=1$ and $w\text{-}\lim_{n\to\infty}x_n=0$ such that $WCS(X)\geq\mathrm{diam}\{x_n\}-\varepsilon$. For any $z\in X$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\limsup_{n\to\infty}\|x_n-z\|=\lim_{n\to\infty}\|x_{n_k}-z\|$. Since X is an Opial space and $w\text{-}\lim_{n\to\infty}x_n=0$, we have

$$\lim_{n\to\infty}\|x_{n_k}-z\| = \liminf_{n\to\infty}\|x_{n_k}-z\| \ge \liminf_{n\to\infty}\|x_{n_k}-0\| = 1.$$

Hence $\inf_{z\in X}\phi(z)=\phi(0)=1$, where $\phi(z)=\limsup_{n\to\infty}\|x_n-z\|$. By the definition of $W(X)$, we have

$$W(X)\le \operatorname{diam}\{x_n\}/\inf_{z\in X}\phi(z)=\operatorname{diam}\{x_n\}\le WCS(X)+\varepsilon.$$

By the arbitrariness of ε, we obtain $W(X)\le WCS(X)$.

Remark. For a Banach space X, if $\beta_C(C)=\beta(C)$ for every closed, bounded, convex subset C, we say X has *the β-invariant property*. The proof of Theorem 4.2.13 shows that, if X is a reflexive Opial space, then X has β-invariant property. It seems that the β-invariant property is related to the ball intersection property (BIP) introduced by Nussbaum in **[Nu-2]**. A Banach space has the BIP if and only if $\beta(\Omega)=\beta_{\overline{\mathrm{B}}_X}(\Omega)$ for every set $\Omega\subseteq\overline{\mathrm{B}}_X$. Indeed every Banach space with the β-invariant property has the BIP, since for every $\Omega\subseteq\overline{\mathrm{B}}_X$,

$$\beta(\Omega)\le\beta_{\overline{\mathrm{B}}_X}(\Omega)=\beta_{\overline{\mathrm{B}}_X}(\overline{\mathrm{co}}\Omega)\le\beta_{\overline{\mathrm{co}}\Omega}(\overline{\mathrm{co}}\Omega)=\beta(\overline{\mathrm{co}}\Omega)=\beta(\Omega).$$

Then every reflexive Opial space has the BIP. However not all spaces with the BIP have the β-invariant property. For example L^∞ spaces have the BIP **[Nu-2]**. But $N_\beta(L^\infty)=1$ (Theorem 3.1.8), then for some closed, bounded, convex and noncompact $C\subseteq L^\infty$, $2\alpha(C)/\beta_C(C)<3/2$, that is $\beta_C(C)>4/3\alpha(C)$. In section 4.4, we will show that $\alpha(\Omega)=\beta(\Omega)$ for any bounded set in L^∞ (Corollary 4.4.6). Hence L^∞ does not have the β-invariant property.

Theorem 4.2.14. *For* $1<p<\infty$, $K_\beta(L^p)=K^0_\beta(L^p)=W(L^p)=\min\{2^{1/p},2^{1-1/p}\}$.

Proof. By Proposition 4.2.11 and Corollary 3.3.6, we have

$$K_\beta(L^p) \ge K_\beta^0(L^p) \ge N_\beta(L^p) = \min\{2^{1/p}, 2^{1-1/p}\}.$$

Also we have $K_\beta(L^p) \ge W(L^p) \ge WCS(L^p) = \min\{2^{1/p}, 2^{1-1/p}\}$ (Theorem 1.2.17). So we can obtain the conclusion by proving $K_\beta(L^p) \le \min\{2^{1/p}, 2^{1-1/p}\}$.

By Theorem 4.2.7, we have $\beta(\Omega) \le 2\alpha(\Omega)/K_\beta(L^p)$ for any bounded set in L^p. In **[Ben-2]**, Benavides showed that there are two bounded sets Ω_1 and Ω_2 in L^p so that $\alpha(\Omega_1) = 2^{1/p}$, $\alpha(\Omega_2) = 2^{1-1/p}$ and $\beta(\Omega_1) = \beta(\Omega_2) = 2$. We repeat his argument below. Hence we obtain that indeed $K_\beta(L^p) \le \min\{2^{1/p}, 2^{1-1/p}\}$.

Next we repeat Benavides' proof. Let $\{A_n\}$ be a sequence of measurable subsets of $[0, 1]$ with $A_i \cap A_j = \emptyset$ if $i \ne j$ and $\mu(A_n) > 0$ for all $n \in \mathbb{N}$, where μ is the Lebesgue measure. For $1 < p < \infty$, $n \in \mathbb{N}$, define $f_{n,p}(x) = (\mu(A_n))^{-1/p}$ if $x \in A_n$ and $f_{n,p}(x) = 0$ otherwise. It is easy to chech that $\|f_{n,p} - f_{m,p}\|_p = 2^{1/p}$ for all $m, n \in \mathbb{N}$ with $m \ne n$ and $\|f_{n,p}\|_p = 1$ for all $n \in \mathbb{N}$. Since for any subsequence $\{f_{n_k,p}\}$ of $\{f_{n,p}\}$, we have $\operatorname{diam}(\{f_{n_k,p}\}) = 2^{1/p}$, $\alpha(\{f_{n,p}\}) = 2^{1/p}$. Next we prove $\beta(\{f_{n,p}\}) = 2$. Let $q \in \mathbb{R}$ be such that $1/p + 1/q = 1$. For any $f \in L^p$, we have

$$\begin{aligned}
&\left|\int_0^1 f_{n,q}(x) f(x)dx\right| = \left|\int_{A_n} f_{n,q}(x) f(x) dx\right| \\
&\le \left(\int_{A_n} |f_{n,q}(x)|^q dx\right)^{1/q} \left(\int_{A_n} |f(x)|^p dx\right)^{1/p} \\
&= \left(\int_{A_n} |f(x)|^p dx\right)^{1/p} \longrightarrow 0 \ (n \longrightarrow \infty),
\end{aligned}$$

since $\mu(A_n) \longrightarrow 0$ and $f \in L^p$. Also we have

$$\begin{aligned}
1 &= \left|\int_0^1 f_{n,p}(x)(\mu(A_n))^{-1/q} dx\right| = \left|\int_0^1 f_{n,p}(x) f_{n,q}(x) dx\right| \\
&\le \left|\int_0^1 (f_{n,p}(x) - f(x)) f_{n,q}(x) dx\right| + \left|\int_0^1 f(x) f_{n,q}(x) dx\right| \\
&\le \|f_{n,p} - f\|_p \|f_{n,q}\|_q + \left|\int_0^1 f(x) f_{n,q}(x) dx\right| \\
&= \|f_{n,p} - f\|_p + \left|\int_0^1 f(x) f_{n,q}(x) dx\right|,
\end{aligned}$$

so $\limsup_{n \to \infty} \|f_{n,p} - f\|_p \ge 1$. Therefore $\beta(\{f_{n,p}\}) \ge 2$. Obviously $\beta(\{f_{n,p}\}) \le 2$, so we have $\beta(\{f_{n,p}\}) = 2$.

Now we consider the "Rademacher functions" defined by $r_0(t)=1$ for all $t\in[0, 1]$ and $r_n(t)=\mathrm{sgn}(\sin 2^n\pi t)$ for $n\in\mathbb{N}$ and $t\in[0, 1]$. It is easy to check that for any $1<p<\infty$, $r_n(t)\in L^p$, $\|r_n\|_p=1$ for any $n\in\mathbb{N}$, and $\|r_n-r_m\|_p=2^{(p-1)/p}$ for all $m, n\in\mathbb{N}$ with $m\neq n$. So $\alpha(\{r_n\})=2^{(p-1)/p}$. And for any $f\in L^p$, it is easy to see that $\int_0^1 r_n(t)f(t)dt \longrightarrow 0$ $(n\longrightarrow\infty)$. Also we have

$$\begin{aligned} 1 = \|r_n\|_2^2 = \int_0^1 r_n^2(t)dt &= \int_0^1 \big(r_n(t)-f(t)\big)r_n(t)dt + \int_0^1 f(t)r_n(t)dt \\ &\leq \|r_n-f\|_p \,\|r_n\|_q + \int_0^1 f(t)r_n(t)dt = \|r_n-f\|_p + \int_0^1 f(t)r_n(t)dt, \end{aligned}$$

so $\limsup_{n\to\infty}\|r_n-f\|_p \geq 1$. Hence $\beta(\{r_n\})=2$.

4.3. $\beta(\Omega)$ and $\delta(\Omega)$ related by $\delta(\overline{\mathrm{B}}_{\mathrm{X}})$ in Banach spaces

We had improved on the inequalities $\beta(\Omega)\leq 2\alpha(\Omega)$ and $\beta(\Omega)\leq 2\delta(\Omega)$ in sections 4.1 and 4.2. In this section, we will give an improvement on $\delta(\Omega)\leq\beta(\Omega)$ by using $\delta(\overline{\mathrm{B}}_{\mathrm{X}})$ in a Banach space X. Also some properties of $\delta(\overline{\mathrm{B}}_{\mathrm{X}})$ are given. In this section all Banach spaces considered are infinite-dimensional.

Lemma 4.3.1. *For any Banach space X,* $\delta(\overline{\mathrm{B}}(x, a))=a\delta(\overline{\mathrm{B}}_{\mathrm{X}})$, where $a\in\mathbb{R}^+$.

Proof. For any $r<\delta(\overline{\mathrm{B}}_{\mathrm{X}})$, there is $\{x_n\}\subseteq\overline{\mathrm{B}}_{\mathrm{X}}$ such that $\|x_n-x_m\|\geq r$ if $n\neq m$. So, for $y_n=ax_n+x\in\overline{\mathrm{B}}(x, a)$, $\|y_n-y_m\|\geq ar$, and hence $\delta(\overline{\mathrm{B}}(x, a))\geq a\delta(\overline{\mathrm{B}}_{\mathrm{X}})$. Conversely, for any $r<\delta(\overline{\mathrm{B}}(x, a))$, there is $\{y_n\}\subseteq\overline{\mathrm{B}}(x, a)$ such that $\|y_n-y_m\|\geq r$ if $m\neq n$. Let $x_n=(y_n-x)/a$, then $\|x_n\|\leq 1$ and $\|x_n-x_m\|=\|y_n-y_m\|/a\geq r/a$. Thus $\delta(\overline{\mathrm{B}}_{\mathrm{X}})\geq\delta(\overline{\mathrm{B}}(x, a))/a$.

Now we can give a connection between $\delta(\Omega)$ and $\beta(\Omega)$ by using $\delta(\overline{\mathrm{B}}_{\mathrm{X}})$.

Theorem 4.3.2. *Let X be a Banach space. Then for any bounded subset Ω of X, we* *have* $\beta(\Omega)\geq 2\delta(\Omega)/\delta(\overline{\mathrm{B}}_{\mathrm{X}})$.

Proof. Let $\beta(\Omega)=b>0$, for any $r>b/2$, there are finitely many balls $\mathrm{B}(x_j, r)$, $j=1, \ldots, n$, such that $\Omega\subseteq \bigcup_{j=1}^{n} \mathrm{B}(x_j, r)$. Then

$$\delta(\Omega)\leq\delta(\bigcup_{j=1}^{n} \mathrm{B}(x_j, r))=\delta(\mathrm{B}(x_1, r))=r\delta(\overline{\mathrm{B}}_{\mathrm{X}}).$$

Since r is arbitrary, $\delta(\Omega)\leq(b/2)\delta(\overline{\mathrm{B}}_{\mathrm{X}})$.

Obviously $\beta(\Omega)\geq 2\alpha(\Omega)/\delta(\overline{\mathrm{B}}_{\mathrm{X}})$ when Ω is an α-minimal set in X. For any infinite-dimensional Banach space X, by Proposition 1.3.7, there is an α-minimal set Ω in X so that $\alpha(\Omega)>0$. Then from Theorem 4.2.7, we have $K_\beta(X)\leq 2\alpha(\Omega)/\beta(\Omega)\leq\delta(\overline{\mathrm{B}}_{\mathrm{X}})$. Hence $\delta(\overline{\mathrm{B}}_{\mathrm{X}})$ is a upper bound for $K_\beta(X)$.

Our next theorem shows that for a class of Banach spaces, $\delta(\overline{\mathrm{B}}_{\mathrm{X}})<2$, so $\beta(\Omega)\geq 2\delta(\Omega)/\delta(\overline{\mathrm{B}}_{\mathrm{X}})$ is indeed a better inequality then $\beta(\Omega)\geq\delta(\Omega)$.

Theorem 4.3.3. *Let X be a Banach space with* $\varepsilon_0(X)<1$, *then* $\delta(\overline{\mathrm{B}}_{\mathrm{X}})<2$.

Proof. By Lemma 1.2.5, $\delta_{\mathrm{X}}(2-)=\lim_{\varepsilon\to 2-}\delta_{\mathrm{X}}(\varepsilon)=1-\varepsilon_0(X)/2$. Let $f(\varepsilon)=\varepsilon-4(1-\delta_{\mathrm{X}}(\varepsilon))$, $f(\varepsilon)$ is continuous on $[0, 2)$, $f(0)<0$ and $f(2-)=2(1-\varepsilon_0(X))>0$. Then there is $a\in(0, 2)$ such that $a=4(1-\delta_{\mathrm{X}}(a))$. We claim $\delta(\overline{\mathrm{B}}_{\mathrm{X}})\leq a$. Otherwise, if $\delta(\overline{\mathrm{B}}_{\mathrm{X}})>a$, there is a sequence $\{x_n\}\subseteq\overline{\mathrm{B}}_{\mathrm{X}}$ such that $\|x_n-x_m\|>a$ if $n\neq m$. Then we have $\|x_n+x_m\|\leq 2(1-\delta_{\mathrm{X}}(a))$ if $m\neq n$. In particular

$$\|x_1-x_2\|\leq\|x_1+x_3\|+\|x_3+x_2\|\leq 4(1-\delta_{\mathrm{X}}(a))=a,$$

a contradiction.

Theorem 4.3.4. *If H is an infinite-dimensional Hilbert space, then* $\delta(\overline{\mathrm{B}}_{\mathrm{H}})=\sqrt{2}$.

Proof. By Theorem 4.2.13, $\delta(\overline{B}_H) \geq K_\beta(H) = \sqrt{2}$. Next we prove $\delta(\overline{B}_H) \leq \sqrt{2}$. Let Ω be any α-minimal subset of $\overline{B}_H$. Since H is reflexive, there is a sequence $\{x_n\} \subseteq \Omega$ with distinct point so that $\{x_n\}$ converges weakly to some x. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $\limsup_{n\to\infty} \|x_n - x\| = \lim_{k\to\infty} \|x_{n_k} - x\|$. Since H is an Opial space, we have

$$\limsup_{n\to\infty} \|x_n - x\| = \lim_{k\to\infty} \|x_{n_k} - x\| \leq \liminf_{k\to\infty} \|x_{n_k} - 0\| \leq 1.$$

Also

$$\|x_n - x_m\|^2 = \|(x_n - x) - (x_m - x)\|^2 = \|x_n - x\|^2 + \|x_m - x\|^2 - 2(x_n - x, x_m - x).$$

So for fixed m, we have

$$\limsup_{n\to\infty} \|x_n - x_m\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|x_m - x\|^2 - 2 \limsup_{n\to\infty} (x_n - x, x_m - x) \leq 1 + \|x_m - x\|^2.$$

Then we obtain

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - x_m\|^2 \leq 2, \quad \text{i.e.,} \quad \limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - x_m\| \leq \sqrt{2}.$$

Hence for any $\varepsilon > 0$, there is a subsequence $\{x_{n_j}\}$ so that $\operatorname{diam}(\{x_{n_j}\}) \leq \sqrt{2} + \varepsilon$. Therefore, $\alpha(\Omega) = \alpha(\{x_{n_j}\}) \leq \sqrt{2} + \varepsilon$. Then $\alpha(\Omega) \leq \sqrt{2}$ since ε is arbitrary. Thus $\delta(\overline{B}_X) \leq \sqrt{2}$ by Lemma 3.2.3.

Next result is helpful to show $\delta(\overline{B}_X) < 2$ for certain spaces with $\varepsilon_0(X) \geq 1$.

Theorem 4.3.5. *Let X and Y are isomorphic Banach spaces, then* $\delta(\overline{B}_Y) \leq d(X, Y)\delta(\overline{B}_X)$.

Proof. Given $\varepsilon > 0$, let $U: Y \to X$ be an isomorphism so that $\|U\| = 1$ and $\|U^{-1}\| \leq d(X, Y) + \varepsilon$, then $U(\overline{B}_Y)$ is a subset of $\overline{B}_X$. For any $0 < r < \delta(\overline{B}_Y)$, there exists a sequence $\{y_n\} \subseteq \overline{B}_Y$ such that $\|y_n - y_m\| \geq r$ if $n \neq m$. Since $\{U(y_n)\} \subseteq \overline{B}_X$, there exist k and l, $k \neq l$, so that $\|U(y_k) - U(y_l)\| \leq \delta(\overline{B}_X) + \varepsilon$. Thus

$$r \le \|U^{-1}U(y_k) - U^{-1}U(y_l)\| \le \|U^{-1}\| \, \|U(y_k) - U(y_l)\| \le (d(X, Y) + \varepsilon)(\delta(\overline{B}_X) + \varepsilon).$$

Since ε and r are arbitrary, the result follows.

Example 4.3.6. Let $l_{p,\infty}$ $(1 < p < \infty)$ be the l^p space renormed by $|x|_\infty = \max\{\|x^+\|, \|x^-\|\}$, where for $x \in l^p$, $x = \{x(n)\}$, $x^+(n) = \max\{x(n), 0\}$ and $x^- = (-x)^+$, $\|\cdot\|$ denotes the usual l^p norm. Then $d(l_{p,\infty}, l^p) \le 2^{1/p}$, and $\delta(\overline{B}_{p,\infty}) \le 2^{2/p}$ $(p > 2)$, where $\overline{B}_{p,\infty}$ denotes the closed unit ball of $l_{p,\infty}$.

In fact, let $U : l^p \longrightarrow l_{p,\infty}$ be the bicontinuous linear operator so that $Ux = x$ for any $x \in l^p$. Then for each $x \in l^p$, we have $|Ux|_\infty = \max\{\|x^+\|, \|x^-\|\} \le \|x\|$, so $\|U\| \le 1$. But for $e_n = \{\delta_{nm} : m = 1, 2, 3, \ldots\}$, where $\delta_{nm} = 0$ if $n \neq m$ and $\delta_{nm} = 1$ if $n = m$, $|Ue_n|_\infty = \|e_n\| = 1$. Hence $\|U\| = 1$.

Also for any $y \in l_{p,\infty}$, we have

$$\|U^{-1}y\| = \|y\| = (\|y^+\|^p + \|y^-\|^p)^{1/p} \le (2|y|_\infty{}^p)^{1/p} = 2^{1/p}|y|_\infty,$$

so $\|U^{-1}\| \le 2^{1/p}$. Let $y_0 = \{1, -1, 0, \ldots, 0, \ldots\}$, then $|y_0|_\infty = 1$ and $\|y_0\| = 2^{1/p}$. Hence we have $\|U^{-1}y_0\| = 2^{1/p}|y_0|_\infty$. Therefore $\|U^{-1}\| = 2^{1/p}$.

Now we obtained $d(l_{p,\infty}, l^p) \le \|U\| \, \|U^{-1}\| = 2^{1/p}$. Then, by Theorems 1.3.2 and 4.3.5, if $p > 2$, $\delta(\overline{B}_{p,\infty}) \le 2^{1/p}\delta(\overline{B}_p) = 2^{2/p} < 2$, where $\overline{B}_p$ denotes the closed unit ball of l^p.

Remark. Bynum [By-1] showed that $l_{p,\infty}$ $(1 < p < \infty)$ lacks normal structure, then $\varepsilon_0(l_{p,\infty}) \ge 1$ since $\varepsilon_0(X) < 1$ implies $N(X) > 1$, that is, X has uniformly normal structure.

From above, we see that $\delta(\overline{B}_X) < 2$ is true for certain spaces, and $\delta(\overline{B}_X)$ can be less than 2 even if X lacks normal structure. Also the exact value of $\delta(\overline{B}_X)$ is given if X is l^p or L^p space (see Chapter 1).

Next we give a relation between $\delta(\overline{B}_X)$ and a packing number which has been calculated in some spaces (see [WelW] and the references therein).

Definition 4.3.7. Let X be a Banach space. We say that *a collection of balls* $\{\overline{B}(x_j, r)\}$ *is packed in* $\overline{B}_X$ provided that:

$$1)\ \|x_j\| \le 1-r \text{ for all } j; \qquad 2)\ \|x_j - x_k\| \ge 2r \text{ for } j \ne k.$$

The *packing number* $\lambda(\overline{B}_X)$ is defined as

$$\sup\{r\text{: infinitely many balls of radius } r \text{ can be packed in } \overline{B}_X\},$$

with $\lambda(\overline{B}_X)=0$ in case infinite packing is impossible.

It seems that it is sometimes easier to calculate $\lambda(\overline{B}_X)$ than $\delta(\overline{B}_X)$. We give the following formula relating $\lambda(\overline{B}_X)$ and $\delta(\overline{B}_X)$. It is a generalization of Theorem 16.7 of [WelW] which gives $1/3 \le \lambda(\overline{B}_X) \le 1/2$.

Theorem 4.3.8. *Let X be an infinite-dimensional Banach space, then* $\lambda(\overline{B}_X)=\delta(\overline{B}_X)/(2+\delta(\overline{B}_X))$.

Proof. First we prove $\lambda(\overline{B}_X) \ge \delta(\overline{B}_X)/(2+\delta(\overline{B}_X))$. For any $r<\delta(\overline{B}_X)$, there is an infinite subset D_r of $\overline{B}_X$ such that $\|x-y\| \ge r$ for any $x, y \in D_r$, $x \ne y$. Let $a=2/(r+2)$, $b=r/(r+2)$, then $\{ay+b\overline{B}_X\colon y \in D_r\}$ is an infinite collection of balls satisfying:

1) for any $y \in D_r$, $\|ay\| \le a = 1-b$; and

2) for any $y_1, y_2 \in D_r$, $y_1 \ne y_2$, $\|ay_1 - ay_2\| \ge ar = 2b$.

So $\lambda(\overline{B}_X) \ge b = r/(r+2)$.

We finish the proof by proving $\delta(\overline{B}_X) \ge 2\lambda(\overline{B}_X)/(1-\lambda(\overline{B}_X))$. For any $r<\lambda(\overline{B}_X)$, there is a sequence $\{x_j\}\subseteq\overline{B}_X$ such that $\|x_j\|\leq 1-r$ for any j, and $\|x_j-x_k\|\geq 2r$ for any j and k, $j\neq k$. Then $\{x_j/(1-r)\}\subseteq\overline{B}_X$ and $\|x_j/(1-r)-x_k/(1-r)\|\geq 2r/(1-r)$ whenever $j\neq k$, so $\delta(\overline{B}_X)\geq 2r/(1-r)$.

Remark. After the author obtained this result, she learned from Dr. Lopez Acedo that it had already been shown by P. L. Papini in [Pa].

Let X be the Banach space gave by James in [Ja], which is not reflexive but is isomorphic with its second conjugate space X^{**}. Kottman [Ko] showed that $\lambda(\overline{B}_X)<1/2$ for this space X. So by Theorem 4.3.8, we have $\delta(\overline{B}_X)<2$.

4.4. Results in classical Banach spaces

As consequences of the results in sections 4.2 and 4.3, we obtain several results for various measures of noncompactness in certain classical spaces. Also connections between the corresponding contractive type mappings are given.

Theorem 4.4.1. *Let H be an infinite-dimensional Hilbert space and let Ω be any bounded subset of H, then $\beta(\Omega)=\sqrt{2}\delta(\Omega)$, and if Ω is α-minimal, $\beta(\Omega)=\sqrt{2}\alpha(\Omega)$.*

Proof. From Theorems 4.2.10 and 4.2.13, we obtain $\beta(\Omega)\leq 2\delta(\Omega)/\sqrt{2}=\sqrt{2}\delta(\Omega)$; and from Theorems 4.3.2 and 4.3.4, we obtain $\beta(\Omega)\geq 2\delta(\Omega)/\sqrt{2}=\sqrt{2}\delta(\Omega)$.

Remark. In [Ben-1], Benavides showed that for any α-minimal sequence $\{x_n\}$ in H, there is a subsequence $\{y_n\}$ of $\{x_n\}$ such that $\beta(\{y_n\})=\sqrt{2}\,\alpha(\{x_n\})$.

Theorem 4.4.2. *Let Ω be any bounded subset of l^p $(1<p<\infty)$, then*

$\beta(\Omega)=2^{1-1/p}\delta(\Omega)$, and if Ω is α-minimal, $\beta(\Omega)=2^{1-1/p}\alpha(\Omega)$. For any bounded subset Ω in l^1, we have $\beta(\Omega)=\alpha(\Omega)=\delta(\Omega)$.

Proof. For $\Omega\subseteq l^p$ $(1<p<\infty)$, by Theorems 4.2.10 and 4.2.13, we have

$$\beta(\Omega)\leq 2\delta(\Omega)/2^{1/p}=2^{1-1/p}\delta(\Omega);$$

by Theorems 4.3.2 and 1.3.2, we have $\beta(\Omega)\geq 2\delta(\Omega)/2^{1/p}=2^{1-1/p}\delta(\Omega)$.

For $\Omega\subseteq l^1$, from Theorems 4.2.6 and 4.2.7, we have $\beta(\Omega)\leq\alpha(\Omega)$. But $\beta(\Omega)\geq\alpha(\Omega)$, so $\beta(\Omega)=\alpha(\Omega)$. Since l^1 is separable, by Propositions 1.3.8 and 1.3.7, there is an α and β-minimal subset Ω' of Ω such that $\beta(\Omega')=\beta(\Omega)$. Hence $\beta(\Omega)=\alpha(\Omega')=\delta(\Omega')\leq\delta(\Omega)\leq\beta(\Omega)$.

Remark. Benavides [Ben-2] proved $\beta(\Omega)=2^{1-1/p}\alpha(\Omega)$ for an α-minimal set Ω in l^p by a direct argument.

Theorem 4.4.3. *Let Ω be a bounded subset of L^p $(1<p<\infty)$, then*

$$\min\{2^{1-1/p}, 2^{1/p}\}\delta(\Omega)\leq\beta(\Omega)\leq\max\{2^{1-1/p}, 2^{1/p}\}\delta(\Omega).$$

And the inequality is true for any α-minimal set Ω if δ is replaced by α.

Proof. Theorems 4.2.10 and 4.2.14 give $\beta(\Omega)\leq 2\delta(\Omega)/\min\{2^{1-1/p}, 2^{1/p}\}$; and Theorems 4.3.2 and 1.3.2 give $\beta(\Omega)\geq 2\delta(\Omega)/\max\{2^{1-1/p}, 2^{1/p}\}$.

Remark. In [BenA], it is also shown that

$$\min\{2^{1-1/p}, 2^{1/p}\}\alpha(\Omega)\leq\beta(\Omega)\leq\max\{2^{1-1/p}, 2^{1/p}\}\alpha(\Omega)$$

for an α-minimal set Ω in L^p.

Next we consider a special class of metric space where $\alpha(\Omega)$ always coincides with $\beta(\Omega)$.

Definition 4.4.4. [AroP] A metric space (M,ρ) will be called *hyperconvex* if for any indexed class of closed balls in M: $\overline{\mathrm{B}}(x_i, r_i)$, $i \in I$, satisfying the condition that $\rho(x_i, x_j) \leq r_i + r_j$ for all i, j in I, the intersection $\bigcap_{i \in I} \overline{\mathrm{B}}(x_i, r_i)$ is not empty.

Propersition 4.4.5. *Let (M, ρ) be a hyperconvex metric space, then for every bounded set Ω in M, $\alpha(\Omega) = \beta(\Omega)$.*

Proof. Let $\alpha(\Omega) = b$. For any $\varepsilon > 0$, there are $\Omega_i \subseteq \Omega$, $i = 1, \ldots, n$, such that $\Omega = \bigcup_{i=1}^{n} \Omega_i$ and $\operatorname{diam}(\Omega_i) < b + \varepsilon$. Let Ω_j $(1 \leq j \leq n)$ such that $\beta(\Omega_j) = \max_{1 \leq i \leq n} \beta(\Omega_i)$, then $\beta(\Omega_j) = \beta(\Omega)$. Let $\operatorname{diam}(\Omega_j) = d$, then the family of closed balls $\{\overline{\mathrm{B}}(x, d/2) : x \in \Omega_j\}$ satisfies $\rho(x, y) \leq d/2 + d/2$ for any $x, y \in \Omega_j$, so the hyperconvexity of (M, ρ) implies that $\bigcap_{x \in \Omega_j} \overline{\mathrm{B}}(x, d/2) \neq \emptyset$. Let $x_0 \in \bigcap_{x \in \Omega_j} \overline{\mathrm{B}}(x, d/2)$, then $\rho(x, x_0) \leq d/2$ for any $x \in \Omega_j$, that is $\Omega_j \subseteq \overline{\mathrm{B}}(x_0, d/2)$. Therefore $\beta(\Omega) = \beta(\Omega_j) \leq d \leq b + \varepsilon$. We obtain $\beta(\Omega) \leq b = \alpha(\Omega)$ by the arbitrariness of ε. As $\alpha(\Omega) \leq \beta(\Omega)$ is always true, we have $\alpha(\Omega) = \beta(\Omega)$.

Hyperconvex spaces are related to Stonian spaces by the theorem of Nachbin–Kelley [Ke], [Na]: The space $C(E)$ of all continuous real functions on a Stonian space (extremally disconnected compact Hausdorff space) with the usual norm is hyperconvex, and every hyperconvex real Banach space is a space $C(E)$ for some Stonian space E. Then l^{∞} and L^{∞} are good examples of hyperconvex spaces [La]. Hence we have the following Corollary:

Corollary 4.4.6. For every bounded set Ω in l^{∞} or L^{∞} space, we have $\alpha(\Omega) = \beta(\Omega)$.

Remark. By a direct proof, [Ben-2] showed that $\alpha(\Omega) = \beta(\Omega)$ in l^{∞} space; [BenA] showed that $\alpha(\Omega)=\beta(\Omega)$ in L^{∞} space. Note that l^{∞} and L^{∞} spaces give examples where $K_{\beta}(X)>N_{\beta}(X)$. By Theorem 4.2.7, we have $K_{\beta}(l^{\infty})=K_{\beta}(L^{\infty})=2$, but $N_{\beta}(l^{\infty})=N_{\beta}(L^{\infty})=1$ (Theorem 3.1.8). Noting that (c_0) is an infinite-dimensional separable, closed subspace of l^{∞}, and $K_{\beta}(c_0)=1$ (Example 4.2.5), we have $K^0_{\beta}(l^{\infty})=1$. Then $K^0_{\beta}(X)$ and $K_{\beta}(X)$ are different.

Next we give some connections between several contractive type mappings.

Theorem 4.4.7. *Let X and Y be two Banach space and let $f: D(f)\subseteq X \longrightarrow Y$. Then*

$$\beta(f)\leq\left(\delta(\overline{\mathrm{B}}_{\mathrm{X}})/K^0_{\beta}(Y)\right)\delta(f)\leq\left(\delta(\overline{\mathrm{B}}_{\mathrm{X}})/K^0_{\beta}(Y)\right)\alpha(f).$$

If Y is separable, we have

$$\beta(f)\leq\left(\delta(\overline{\mathrm{B}}_{\mathrm{X}})/K_{\beta}(Y)\right)\delta(f)\leq\left(\delta(\overline{\mathrm{B}}_{\mathrm{X}})/K_{\beta}(Y)\right)\alpha(f).$$

Proof. By Theorem 4.1.6, we have $\delta(f)\leq\alpha(f)$. Let Ω be a bounded subset of $D(f)$, from Theorems 4.2.10 and 4.3.2, we obtain

$$\begin{aligned}\beta(f(\Omega))&\leq 2\delta(f(\Omega))/K^0_{\beta}(Y)\leq 2\delta(f)\delta(\Omega)/K^0_{\beta}(Y)\\&\leq\delta(f)\delta(\overline{\mathrm{B}}_{\mathrm{X}})\beta(\Omega)/K^0_{\beta}(Y)=\left(\delta(\overline{\mathrm{B}}_{\mathrm{X}})/K^0_{\beta}(Y)\right)\delta(f)\beta(\Omega).\end{aligned}$$

Theorem 4.4.8. *Let X and Y be two Banach spaces and let $f: D(f)\subseteq X\longrightarrow Y$. Then $\delta(f)\leq\alpha(f)\leq\left(2/K_{\beta}(X)\right)\beta(f)$.*

Proof. For any bounded subset Ω of $D(f)$, by Theorem 4.2.7, we have

$$\alpha(f(\Omega))\leq\beta(f(\Omega))\leq\beta(f)\beta(\Omega)\leq\beta(f)\left(2/K_{\beta}(X)\right)\alpha(\Omega).$$

In case X and Y are Hilbert or l^p, L^p spaces, we have the following corollaries.

Corollary 4.4.9. *Let H_1 and H_2 be two Hilbert spaces and let $f: D(f)\subseteq H_1\longrightarrow H_2$. Then $\beta(f)\leq\delta(f)\leq\alpha(f)\leq\sqrt{2}\beta(f)$.*

Remark. If H is a separable Hilbert space, and $f: D(f) \subseteq H \to H$ is a k-set contraction, Benavides [Ben-1] showed that $\beta(f) \leq \alpha(f)$.

Corollary 4.4.10. 1) *For* $f: D(f) \subseteq l^p \to l^r$ $(1 \leq p, r < \infty)$, *we have*

$$\beta(f) \leq 2^{1/p-1/r}\delta(f) \leq 2^{1/p-1/r}\alpha(f) \quad \textit{and} \quad \delta(f) \leq \alpha(f) \leq 2^{1-1/p}\beta(f).$$

2) *For* $f: D(f) \subseteq l^p \to l^\infty$ $(1 \leq p < \infty)$, *we have*

$$\beta(f) \leq 2^{1/p}\delta(f) \leq 2^{1/p}\alpha(f) \quad \textit{and} \quad \delta(f) \leq \alpha(f) \leq 2^{1-1/p}\beta(f).$$

3) *For* $f: D(f) \subseteq l^\infty \to l^p$ $(1 \leq p < \infty)$, *we have*

$$\beta(f) \leq 2^{1-1/p}\delta(f) \leq 2^{1-1/p}\alpha(f) \quad \textit{and} \quad \delta(f) \leq \alpha(f) \leq \beta(f).$$

4) *For* $f: D(f) \subseteq l^\infty \to l^\infty$, *we have* $\delta(f) \leq \alpha(f) = \beta(f)$. *For* $f: D(f) \subseteq l^1 \to l^1$, *we have* $\delta(f) = \alpha(f) = \beta(f)$.

Remark. Benavides [Ben-2] proved that $\beta(f) \leq \alpha(f)$ if $f: D(f) \subseteq l^p \to l^p$ $(1 < p < \infty)$ is a k-set contraction, and $\beta(f) = \alpha(f)$ in l^1 and l^∞.

Corollary 4.4.11. *For* $f: D(f) \subseteq L^p \to L^r$ $(1 < p, r < \infty)$, *we have*

$$\beta(f) \leq a(p, r)\delta(f) \leq a(p, r)\alpha(f),$$

where

$$a(p, r) = \max\{2^{1-1/p}, 2^{1/p}\}/\min\{2^{1-1/r}, 2^{1/r}\};$$

and $\delta(f) \leq \alpha(f) \leq \max\{2^{1-1/p}, 2^{1/p}\}\beta(f)$.

Remark. For $f: D(f) \subseteq L^p \to L^p$ $(1 < p < \infty)$, it is shown that $\beta(f) \leq 2^{|1-2/p|}\alpha(f)$ in [BenA]. Our Corollary 4.4.11 contains this result as a special case.

CHAPTER FIVE

SEMINORMS OF LINEAR OPERATORS

In order to estimate the departure from compactness of a linear operator $T \in \mathscr{L}(X, Y)$, investigations on seminorms of T are helpful. Several seminorms, such as $\alpha(T)$, $\beta(T)$ and $\|T\|_{\mathscr{K}}$ have been studied in recent years. Certain of their properties and some relations between them have been obtained. We mentioned some of these in Chapter one. [GolM] showed that $\beta(T)/2 \leq \beta(T^*) \leq 2\beta(T)$; [Nu-2] showed that $\alpha(T) \leq \beta(T^*)$ and $\alpha(T^*) \leq \beta(T)$; [Web-1] and [Web-2] showed that $\alpha(T) = \beta(T)$ and $\beta(T^*) \leq \beta(T)$ in Hilbert spaces. We continue these works and relate $\beta(T)$ and $\beta(T^*)$ by using a geometrical number $R_\beta(X)$. For $T \in \mathscr{L}(X, Y)$, we will show that $\beta(T)/R_\beta(X) \leq \beta(T^*) \leq R_\beta(Y^*)\beta(T)$. It is easy to see that if X is uniformly convex or uniformly smooth, $R_\beta(X) < 2$. We are able to prove that $R_\beta(X) = 1$ in certain spaces such as Hilbert or l^p $(1 < p < \infty)$ spaces. Hence our results extend the inequalities $\beta(T)/2 \leq \beta(T^*) \leq 2\beta(T)$. We also obtain estimates for $R_\beta(X)$ in terms of $N_\beta(X)$ and other geometrical coefficients in X.

Another seminorm related to $\beta(T)$ is also investigated.

For $T \in \mathscr{L}(X)$, we show by an elementary argument that $I - T$ has finite ascent and finite descent if $\alpha(T) < 1$ or $\beta(T) < 1$. For any $\lambda \in \mathbb{R}$, if $|\lambda| > \alpha(T)$, we prove that λ in the resolvent set of T or λ is an eigenvalue of T of finite algebraic multiplicity. Also for $r > \alpha(T)$, there are at most finitely many points λ in the spectrum of T with $|\lambda| \geq r$ and all such λ's are eigenvalues of T. This answers the question posed by Martin in [Mar] who needed to assume $r > 2\alpha(T)$ in proving a similar fact.

This chapter includes part of the work of [Z-2].

5.1. $\beta(T)$ and $\beta(T^*)$ related by a geometrical coefficient

We intend to give an inequality between $\beta(T)$ and $\beta(T^*)$ which is sharper than $\beta(T)/2 \leq \beta(T^*) \leq 2\beta(T)$ and implies $\beta(T)=\beta(T^*)$ in certain spaces. Our inequality use the following geometrical number.

Definition 5.1.1. Let X be a Banach space. We define a geometrical coefficient $R_\beta(X)$ in X by:

$$R_\beta(X)=\sup\left\{\beta_C(C)/2 : C \subseteq \overline{\mathrm{B}}_X \text{ is closed and convex}\right\}.$$

We have $1 \leq R_\beta(X) \leq 2$ since $\beta_{\overline{\mathrm{B}}_X}(\overline{\mathrm{B}}_X)=2$ and $\beta_C(C) \leq 2\mathrm{diam}(C) \leq 4$. Later we will see that $R_\beta(X)<2$ in several classes of Banach spaces, in particular, $R_\beta(X)=1$ in certain spaces.

We can relate $\beta(T)$ and $\beta(T^*)$ by using $R_\beta(X)$ as follows.

Theorem 5.1.2. *Let X and Y be Banach spaces. For every $T \in \mathscr{L}(X, Y)$, we have* $\beta(T) \leq R_\beta(X)\beta(T^*)$ and $\beta(T^*) \leq R_\beta(Y^*)\beta(T)$.

In order to deduce the second inequality in Theorem 5.1.2 from the first one, we obtain $\beta(T^*) \leq R_\beta(Y^*)\beta(T^{**})$, so if $\beta(T^{**}) \leq \beta(T)$, the proof follows. Sedaev [Se] proved $\beta(T^{**}) \leq \beta(T)$ for $T \in \mathscr{L}(X)$, and his proof remains valid for $T \in \mathscr{L}(X, Y)$. Here we give a proof since his paper is only available in Russian.

Theorem 5.1.3. *Let X and Y be Banach spaces and let $T \in \mathscr{L}(X, Y)$. Then* $\beta(T^{**}) \leq \beta(T)$.

Proof. For any $a>\beta(T)=\beta(T(\overline{B}_X))/2$, there are $y_1, \ldots, y_n \in Y$ so that

$$T(\overline{B}_X) \subseteq \bigcup_{i=1}^{n} B(y_i, a),$$

thus

$$J_Y T(\overline{B}_X) \subseteq \bigcup_{i=1}^{n} J_Y \overline{B}(y_i, a) = \bigcup_{i=1}^{n} (J_Y y_i + a J_Y \overline{B}_Y).$$

For $\Omega \subseteq Y^{**}$, we use w*-cl(Ω) to denote the weak* closure of Ω. Note that for any Banach space X, $w^*\text{-cl}(J_X \overline{B}_X) = \overline{B}_X^{**}$, where $\overline{B}_X^{**}$ be the closed unit ball of X^{**} (see [TL], p.177, Theorem 10.7). We have

$$w^*\text{-cl}\big(J_Y T(\overline{B}_X)\big) \subseteq w^*\text{-cl}\Big(\bigcup_{i=1}^{n} (J_Y y_i + a J_Y \overline{B}_Y)\Big) = \bigcup_{i=1}^{n} (J_Y y_i + a \overline{B}_Y^{**}),$$

where $\overline{B}_Y^{**}$ denotes the closed unit ball in Y^{**}. Let $x^{**} \in \overline{B}_X^{**}$. For any $\varepsilon>0$ and any finitely many $y_1^*, \ldots, y_m^* \in Y^*$, there is $x \in J_X \overline{B}_X$ such that $|(x^{**}-x, T^* y_i^*)| < \varepsilon$ for all $i=1, 2, \ldots, m$, that is $|(T^{**}x^{**} - T^{**}x,\ y_i^*)| < \varepsilon$. Hence

$$T^{**}(\overline{B}_X^{**}) \subseteq w^*\text{-cl}\big(T^{**}(J_X \overline{B}_X)\big) = w^*\text{-cl}\big(J_Y T(\overline{B}_X)\big)$$

since $J_Y T = T^{**} J_X$. Therefore we obtain $2\beta(T^{**}) = \beta(T^{**}(\overline{B}_X^{**})) \le 2a$. $\beta(T^{**}) \le \beta(T)$ follows by letting $a \longrightarrow \beta(T)$.

Proof of Theorem 5.1.2. We only need to prove the first inequality as shown above. Let $k=\beta(T^*)$. Since $\beta(T)=\beta(T(\overline{B}_X))/2$, the first inequality is equivalent to $\beta(T(\overline{B}_X)) \le 2kR_\beta(X)$.

Let $\overline{B}_Y^*$ be the closed unit ball of Y^*. For any $\varepsilon>0$, there are $x_j^* \in X^*$, $j=1, \ldots, N$, such that

$$T^*(\overline{B}_Y^*) \subseteq \bigcup_{j=1}^{N} B(x_j^*, k+\varepsilon).$$

For any fixed $j: 1 \le j \le N$, $\{(x_j^*, x):\ x \in \overline{B}_X\}$ is bounded set in $\mathbb{R}$, so it is precompact. There are finitely many closed intervals S_l^j $(1 \le l \le M_j)$ of length $\le \varepsilon$, such that

$$\{(x_j^*, x) : x \in \overline{\mathrm{B}}_X\} \subseteq \bigcup_{l=1}^{M_j} S_l^j .$$

Let

$$\Lambda = \{p: p=(p_1, \ldots, p_N),\ p_j \in \{1, \ldots, M_j\},\ j=1, \ldots, N\},$$

a finite set. For any $p \in \Lambda$, $p=(p_1, \ldots, p_N)$, let

$$E_p = \{x \in \overline{\mathrm{B}}_X : (x_j^*, x) \in S_{p_j}^j,\ 1 \leq j \leq N\}.$$

Then E_p is closed and convex, and $\overline{\mathrm{B}}_X = \bigcup_{p \in \Lambda} E_p$.

As $\beta_{E_p}(E_p) \leq 2R_\beta(X)$, there are $z_{p1}, z_{p2}, \ldots, z_{p\,k(p)} \in E_p$, such that

$$E_p \subseteq \bigcup_{i=1}^{k(p)} \mathrm{B}(z_{pi}, R_\beta(X)+\varepsilon).$$

For any $y_p \in E_p$, there is a z_{pi} such that $\|y_p - z_{pi}\| \leq R_\beta(X)+\varepsilon$. By the Hahn-Banach theorem, there exists $y^* \in Y^*$, $\|y^*\|=1$ such that $\|Ty_p - Tz_{pi}\| = (y^*, Ty_p - Tz_{pi})$. For this y^*, there is x_j^* such that $\|T^*y^* - x_j^*\| \leq k+\varepsilon$. Noting that $(x_j^*, y_p) \in S_{p_j}^j$ and $(x_j^*, z_{pi}) \in S_{p_j}^j$, we have $|(x_j^*, y_p - z_{pi})| = |(x_j^*, y_p) - (x_j^*, z_{pi})| \leq \varepsilon$. Then

$$\begin{aligned}
\|Ty_p - Tz_{pi}\| &= (T^*y^*, y_p - z_{pi}) \\
&= (T^*y^* - x_j^*, y_p - z_{pi}) + (x_j^*, y_p - z_{pi}) \\
&\leq \|T^*y^* - x_j^*\| \, \|y_p - z_{pi}\| + |(x_j^*, y_p - z_{pi})| \\
&\leq (k+\varepsilon)\big(R_\beta(X)+\varepsilon\big) + \varepsilon = kR_\beta(X) + \big(R_\beta(X)+k+1+\varepsilon\big)\varepsilon.
\end{aligned}$$

Hence, $T(E_p)$ can be covered by a finite number of balls of radius at most $kR_\beta(X) + \big(R_\beta(X)+k+1+\varepsilon\big)\varepsilon$. Since $T(\overline{\mathrm{B}}_X) = \bigcup_{p \in \Lambda} T(E_p)$ and Λ is a finite set, $T(\overline{\mathrm{B}}_X)$ can also be covered by finitely many balls of radius at most $kR_\beta(X) + \big(R_\beta(X)+k+1+\varepsilon\big)\varepsilon$. This proves $\beta(T(\overline{\mathrm{B}}_X)) \leq 2kR_\beta(X)$, as ε is arbitrary.

As a consequence of Theorem 5.1.2, using Nussbaum's result $\alpha(T)\leq\beta(T^*)$ (Theorem 1.3.12), we obtain the following connection between $\alpha(T)$ and $\beta(T)$.

Corollary 5.1.4. *Let X and Y be Banach spaces. For every $T\in\mathscr{L}(X,Y)$, we have* $\alpha(T)\leq R_{\beta}(Y^*)\beta(T)$.

We will show that a class of spaces satisfies $R_{\beta}(X)=1$. First we give the following definition.

Definition 5.1.5. A Banach space X is said to satisfy the *U-condition* if for any sequence $\{x_n\}\subseteq\overline{B}_X$, it follows that

$$\inf_{x\in\overline{co}\{x_n\}}\left\{\liminf_{n\to\infty}\|x_n-x\|\right\}\leq 1.$$

Any reflexive Opial Banach space X satisfies the U-condition. In fact, for any sequence $\{x_n\}\subseteq\overline{B}_X$, there is a subsequence $\{x_{n_k}\}$ that converges weakly to some $x_0\in\overline{co}\{x_n\}$. Then

$$\liminf_{n\to\infty}\|x_n-x_0\|\leq\liminf_{k\to\infty}\|x_{n_k}-x_0\|\leq\liminf_{k\to\infty}\|x_{n_k}-0\|\leq 1.$$

Hence Hilbert and l^p $(1<p<\infty)$ spaces satisfy the U-condition.

Theorem 5.1.6. *If X is a Banach space satisfing the U-condition, then* $R_{\beta}(X)=1$.

Proof. We prove that for any closed, convex subset C of $\overline{B}_X$, $\beta_C(C)\leq 2$. Suppose $\beta_C(C)=2a>0$. Then for any $\varepsilon: 0<\varepsilon<a$, there is a sequence $\{x_n\}\subseteq C$ such that $\beta_D(D)\geq 2(a-\varepsilon)$, where $D=\overline{co}\{x_n\}$ (Lemma 3.2.4). By Proposition 1.3.8, there is a β-minimal subset Ω of D such that $\beta_D(D)=\beta_D(\Omega)$. Let $\{z_n\}\subseteq\Omega$ be such that $z_n\neq z_m$ whenever $n\neq m$. Since X satisfies the U-condition, we have

$$\inf_{z\in\overline{\mathrm{co}}\{z_n\}}\left\{\liminf_{n\to\infty}\|z_n-z\|\right\}\le 1.$$

There exists $z\in\overline{\mathrm{co}}\{z_n\}\subseteq D$ such that $\liminf_{n\to\infty}\|z_n-z\|\le 1+\varepsilon$, so there is a subsequence $\{z_{n_k}\}$ such that $\lim_{k\to\infty}\|z_{n_k}-z\|\le 1+\varepsilon$. Then

$$\begin{aligned}\beta_C(C)&\le\beta_D(\Omega)+2\varepsilon=\beta_D(\{z_n\})+2\varepsilon\\&=\beta_D(\{z_{n_k}\})+2\varepsilon\le 2\lim_{k\to\infty}\|z_{n_k}-z\|+2\varepsilon\le 2+4\varepsilon\end{aligned}$$

Letting $\varepsilon\to 0$, we obtain $\beta_C(C)\le 2$.

From Theorems 5.1.2 and 5.1.6 and Corollary 5.1.4, the following result is immediate.

Corollary 5.1.7. *Let X and Y be Banach spaces and let $T\in\mathscr{L}(X,Y)$. If X satisfies the U-condition, then $\beta(T)\le\beta(T^*)$; if Y^* satisfies the U-condition, then $\beta(T^*)\le\beta(T)$ and $\alpha(T)\le\beta(T)$. In particular, if $T\in\mathscr{L}(H_1,H_2)$, where H_1 and H_2 are Hilbert spaces, or $T\in\mathscr{L}(l^p,l^r)$ $(1<p,r<\infty)$, $\beta(T^*)=\beta(T)$.*

Remark. For $T\in\mathscr{L}(H_1,H_2)$, results of Webb [Web-1], [Web-2] (Theorem 1.3.14) and Nussbaum [N-2] (Theorem 1.3.12) show that $\beta(T^*)=\beta(T)$ directly (see Chapter one). Also for $T\in\mathscr{L}(l^p)$ $(1<p<\infty)$, we know $T^*\in\mathscr{L}(l^q)$ $(1/p+1/q=1)$, Benavides [Ben-2] showed that $\beta(T)\le\alpha(T)$ and $\beta(T^*)\le\alpha(T^*)$ (see also Corollary 4.4.10). By these and $\alpha(T)\le\beta(T^*)$, $\alpha(T^*)\le\beta(T)$ (Nussbaum, Theorem 1.3.12), one obtains that $\alpha(T)=\beta(T)=\beta(T^*)$ holds throughout. [GolM] proved $\beta(T^*)=\beta(T)$ for $T\in\mathscr{L}(l^p,l^r)$ $(1<p,r<\infty)$ or $T\in\mathscr{L}(c_0)$ by using different methods.

Example 5.1.8. $R_\beta(l^1)=R_\beta(l^\infty)=2$.

For l^∞, let $x_n=\{x_{nm}: m=1,2,3,\dots\}$, where $x_{nm}=-1$ if $1\le m\le n$, $x_{nm}=1$

if $m>n$. Then $x_n \in l^\infty$, and $\|x_n\|_\infty = 1$. It is easy to verify that $\operatorname{dist}(x_{n+1}, \operatorname{co}\{x_i\}_1^n) = 2$. Let $C = \overline{\operatorname{co}}\{x_n\}$, then $\beta_C(C) = 4$. In fact, if $\beta_C(C) := 2a < 4$, there is $\varepsilon > 0$ such that $a + 2\varepsilon < 2$. There exist $y_1, \ldots, y_k \in C$ such that $C \subseteq \bigcup_{j=1}^{k} B(y_j, a+\varepsilon)$. Some $B(y_j, a+\varepsilon)$ $(1 \le j \le k)$ contains a subsequence of $\{x_n\}$. For this y_j, there is $z \in \operatorname{co}\{x_i\}_1^p$ $(p \in \mathbb{N})$ so that $\|y_j - z\| < \varepsilon$. Then for some $n > p$ so that $x_{n+1} \in B(y_j, a+\varepsilon)$, since $z \in \operatorname{co}\{x_i\}_1^n$ we have

$$\operatorname{dist}(x_{n+1}, \operatorname{co}\{x_i\}_1^n) \le \|x_{n+1} - z\| \le \|x_{n+1} - y_j\| + \|y_j - z\| < a + 2\varepsilon < 2.$$

This is a contradiction. Thus $R_\beta(l^\infty) = 2$.

For l^1, consider $e_n = \{\delta_{nm} : m = 1, 2, 3, \ldots\}$, where $\delta_{nm} = 1$ if $m = n$, $\delta_{nm} = 0$ if $m \ne n$, we have $\operatorname{dist}(e_{n+1}, \operatorname{co}\{e_i\}_1^n) = 2$. Let $C = \overline{\operatorname{co}}\{e_n\}$, then $\beta_C(C) = 4$. Therefore $R_\beta(l^1) = 2$.

For $T \in \mathscr{L}(l^1)$ or $T \in \mathscr{L}(l^\infty)$, from the example above, we can not obtain a better result than $\beta(T) \le 2\beta(T^*)$ by using Theorem 5.1.2. However by the results in Benavides [Ben-2] (see also Corollary 4.4.10), we have $\beta(T) = \alpha(T)$. Combining this with $\alpha(T) \le \beta(T^*)$, we obtain $\beta(T) \le \beta(T^*)$. In particular for $T \in \mathscr{L}(l^1)$, we also have $\beta(T^*) \le \beta(T^{**}) \le \beta(T)$ as $T^* \in \mathscr{L}(l^\infty)$, hence $\beta(T) = \beta(T^*)$. So it is necessary to give the following result.

Theorem 5.1.9. *Let X and Y be Banach spaces and let $T \in \mathscr{L}(X, Y)$. Then we have $\beta(T) \le 2\beta(T^*)/K_\beta(Y)$ and $\beta(T^*) \le 2\beta(T)/K_\beta(X^*)$.*

Proof. By Theorems 4.2.7 and 1.3.12, we have

$$\beta(T) = \beta(T\overline{B}_X)/2 \le \alpha(T\overline{B}_X)/K_\beta(Y) \le 2\alpha(T)/K_\beta(Y) \le 2\beta(T^*)/K_\beta(Y).$$

Also $\beta(T^*) \le 2\beta(T^{**})/K_\beta(X^*) \le 2\beta(T)/K_\beta(X^*)$ (Theorem 5.1.3).

As a consequence of Corollaries 4.4.10 and 5.1.7, we obtain the following classical result of H. R. Pitt on compact linear operators [LinT-2], p.31.

Corollary 5.1.10. *Let* $T \in \mathscr{L}(l^p, l^r)$, *where* $1<r<p<\infty$, *then* $\beta(T)=0$, *that is,* T *is compact.*

Proof. By Corollary 4.4.10, $\beta(T) \leq 2^{1/p-1/r}\alpha(T)$, and by Corollary 5.1.7, $\alpha(T) \leq \beta(T)$, thus $(1-2^{1/p-1/r})\beta(T) \leq 0$. Therefore $\beta(T)=0$

If $T \in \mathscr{L}(l^p, l^1)$ $(2<p<\infty)$, we can also prove $\beta(T)=0$. In fact, by Corollary 4.4.10, we have $\beta(T) \leq 2^{1/p-1}\alpha(T)$. Also by Theorem 5.1.9, we have $\beta(T^*) \leq 2\beta(T)/2^{1/q}$, where $1/q+1/p=1$. Noting that $\alpha(T) \leq \beta(T^*)$ (Theorem 1.3.12), we obtain $\beta(T) \leq 2^{1/p-1/q}\beta(T)$. Hence $\beta(T)=0$.

Next we show that there are other spaces X and Y such that $\mathscr{L}(X, Y)$ and $\mathscr{K}(X, Y)$ are equal.

Corollary 5.1.11. *Let* X *be a Banach space and* H *be a Hilbert space. If* $\delta(\overline{\mathrm{B}}_X) < \sqrt{2}$, *then for any* $T \in \mathscr{L}(X, H)$, $\beta(T)=0$, *that is,* T *is compact.*

Proof. By Theorem 4.4.7 and Corollary 5.1.4, we have

$$\beta(T) \leq \big(\delta(\overline{\mathrm{B}}_X)/K^0_\beta(H)\big)\alpha(T) \leq \big(\delta(\overline{\mathrm{B}}_X)/K^0_\beta(H)\big)R_\beta(H^*)\beta(T).$$

Since $\delta(\overline{\mathrm{B}}_X) < \sqrt{2} = K^0_\beta(H)$ (Theorem 4.2.13) and $R_\beta(H^*)=1$ (Theorem 5.1.6), we obtain $\beta(T)=0$.

Note that $\delta(\overline{\mathrm{B}}_X) < \sqrt{2}$ in finite-dimensional spaces and for $X=l^p$, $p>2$ and $X=l_{p,\infty}$, $p>4$ (see Example 4.3.6 in Chapter 4).

The following result gives an upper bound for $R_\beta(X)$.

Proposition 5.1.12. *For any Banach space X,* $R_\beta(X) \le \delta(\overline{B}_X)/N_\beta(X)$. *In particular,* $R_\beta(L^p) \le 2^{|1-2/p|}$ $(1<p<\infty)$.

Proof. For any closed convex subset C of $\overline{B}_X$, Theorem 3.2.5 shows that

$$\beta_C(C) \le 2\delta(C)/N_\beta(X) \le 2\delta(\overline{B}_X)/N_\beta(X).$$

For L^p, By Corollary 3.3.6, we have $N_\beta(L^p) = \min\{2^{1/p}, 2^{1-1/p}\}$; and by Theorem 1.3.2, we have $\delta(\overline{B}^p) = \max\{2^{1/p}, 2^{1-1/p}\}$. Hence $R_\beta(L^p) \le 2^{1-2/p}$ if $2<p<\infty$; $R_\beta(L^p) \le 2^{2/p-1}$ if $1<p\le 2$.

From proposition 5.1.12, we see that if $\delta(\overline{B}_X)<2$ or $N_\beta(X)>1$, then $R_\beta(X)<2$. In particular, if X is uniformly convex, $R_\beta(X)<2$. Also if X is uniformly smooth, $N_\beta(X)>1$ since $N(X)>1$ (cf. **[GoeK-1]**, Theorem 14.3), so $R_\beta(X)<2$.

$\delta(\overline{B}_X)$ can be less than 2 in some nonreflexive Banach spaces or spaces without normal structure (see Chapter 4), but $N_\beta(X)=1$ for those spaces (see Chapter 3). Also in some spaces X with $N_\beta(X)>1$, $\delta(\overline{B}_X)$ can be 2. Let $l_{p,1}$ $(1<p<\infty)$ be the l^p space renormed by $|x|_1 = \|x^+\| + \|x^-\|$, where for $x \in l^p$, x^+, x^- are as those in Example 4.3.6, and $\|.\|$ denotes the usual l^p norm. $N_\beta(l_{p,1}) = WCS(l_{p,1}) = 2^{1/p}$ **[By-2]**, but we have $\delta(\overline{B}_{p,1})=2$. In fact, let $e_n = \{\delta_{nm} : m=1, 2, 3, \ldots\}$, where $\delta_{nm}=1$ if $m=n$ and $\delta_{nm}=0$ if $m \ne n$. Since $|e_n|_1=1$ and $|e_n-e_k|_1=2$ wherever $n \ne k$, we see that $\delta(\overline{B}_{p,1})=2$.

From Theorem 5.1.2 and Proposition 5.1.12, we have

Corollary 5.1.13. *For* $T \in \mathscr{L}(L^p, L^r)$ $(1<p, r<\infty)$, *we have*

$$2^{-|1-2/p|}\beta(T) \le \beta(T^*) \le 2^{|1-2/r|}\beta(T)$$

and

$$\alpha(T) \le 2^{|1-2/r|}\beta(T).$$

Remark. For $T\in\mathscr{L}(L^P)$ $(1<p<\infty)$, [BenA] proved $\beta(T)\le 2^{|1-2/p|}\alpha(T)$ (see also Corollary 4.4.11). When $r=p$, Corollary 5.1.13 can be deduced from this inequality and the facts $\alpha(T)\le\beta(T^*)$, $\alpha(T^*)\le\beta(T)$.

The estimation $R_\beta(X)\le\delta(\overline{\mathrm{B}}_X)/N_\beta(X)$ is sharp for some spaces, for example in l^p $(1<p<\infty)$ spaces, $\delta(\overline{\mathrm{B}}_p)=N_\beta(l^p)=2^{1/p}$. But this is not so for (c_0). In (c_0), it is easy to verify that $\delta(\overline{\mathrm{B}}_1)=2$ ($\overline{\mathrm{B}}_1$ is the closed unit ball of (c_0)) by considering the sequence $x_n=\{x_{nm}: m=1, 2, 3, \ldots\}$, where $x_{nm}=-1$ if $m\le n$, $x_{nm}=1$ if $m=n+1$, $x_{nm}=0$ if $m>n+1$. Also $N_\beta(c_0)=1$ (Theorem 3.1.8). The following example shows that $R_\beta(c_0)$ is not 2.

Example 5.1.14. $R_\beta(c_0)\le 4/3$.

If $R_\beta(c_0)=a>4/3$, for any ε: $0<\varepsilon<\min\{(a-1)/2, (3a-4)/(3+a)\}$, there is $C\subseteq\overline{\mathrm{B}}_1$ such that $\beta_C(C)>2(a-\varepsilon)$. From the proof of Lemma 3.2.4, there exists a sequence $\{x_n\}\subseteq C$ such that $\mathrm{dist}(x_{n+1}, \mathrm{co}\{x_i\}_1^n)>a-\varepsilon$. Let $x_n=\{x_{ni}: i=1, 2, 3,\ldots\}$. Since $x_{1i}\longrightarrow 0$ $(i\longrightarrow\infty)$, there is N such that $|x_{1i}|<\varepsilon$ whenever $i>N$. There exists a subsequence $\{n_k\}$ of $\{n\}$ such that $\{x_{n_k,i}\}$ converges for every $1\le i\le N$. There is a K so that $|x_{n_k,i}-x_{n_j,i}|<\varepsilon^2/(1-\varepsilon)$ for any k, $j>K$ and $1\le i\le N$. Let $\lambda=(2-a+\varepsilon)/(1-\varepsilon)$. Since $a\le 2$ and $\varepsilon<(a-1)/2$, we know $0<\lambda<1$. Since $\varepsilon<(3a-4)/(3+a)$, $4-2a+2\varepsilon<a-a\varepsilon-\varepsilon$. Hence we have

$$\begin{aligned}2\lambda&=(\varepsilon^2+4-2a+2\varepsilon)/(1-\varepsilon)-\varepsilon^2/(1-\varepsilon)\\&<\big(\varepsilon(\varepsilon-1)+a(1-\varepsilon)\big)/(1-\varepsilon)-\varepsilon^2/(1-\varepsilon)=(a-\varepsilon)-\varepsilon^2/(1-\varepsilon).\end{aligned}$$

For any $k>K+1$, let j be such that $k>j>K$. If $1\le i\le N$, we obtain

$$
\begin{aligned}
&|x_{n_k,i} - \lambda x_{1i} - (1-\lambda)x_{n_j,i}| \\
&\le |x_{n_k,i} - x_{n_j,i}| + \lambda|x_{n_j,i} - x_{1i}| \\
&< \varepsilon^2/(1-\varepsilon) + 2\lambda < a-\varepsilon;
\end{aligned}
$$

and if $i > N$, we obtain

$$|x_{n_k,i} - \lambda x_{1i} - (1-\lambda)x_{n_j,i}| \le 1+\lambda\varepsilon+(1-\lambda)=2-\lambda(1-\varepsilon)=a-\varepsilon.$$

This is a contradiction to $\mathrm{dist}(x_{n+1}, \mathrm{co}\{x_i\}_1^n) > a-\varepsilon$.

Remark. It seems likely that the value of $R_\beta(c_0)$ should be 1, but we can only obtain the above result.

We intend to establish another upper bound for $R_\beta(X)$, the following definition given in [BenL-2] is needed.

Definition 5.1.15. Let X be a Banach space. *A modulus of noncompact convexity* $S_X(\varepsilon)$ $(0 \le \varepsilon < \delta(\overline{B}_X))$ is defined as:

$$S_X(\varepsilon) = \inf\Big\{1 - \inf_{z\in C}\|z\| : C \subseteq \overline{B}_X \text{ is closed and convex, } \delta(C) > \varepsilon\Big\}.$$

Obviously, $\delta_X(\varepsilon) \le S_X(\varepsilon)$. In certain spaces the inequality holds, for example in l^p $(1<p<\infty)$ spaces, $\delta_{l^p}(\varepsilon) < S_{l^p}(\varepsilon) = 1-(1-\varepsilon^p/2)^{1/p}$ [BenL-2]. Now we can give another upper bound for $R_\beta(X)$ by using $S_X(\varepsilon)$ and $N_\beta(X)$.

Proposition 5.1.16. *Let X be a Banach space. If $N_\beta(X)=\delta(\overline{B}_X)$, then $R_\beta(X)=1$; if $N_\beta(X)<\delta(\overline{B}_X)$, then $R_\beta(X)\le 2-S_X(N_\beta(X))$.*

Proof. Let C be any closed, convex subset of $\overline{B}_X$. If $\beta_C(C)>2$, for any $\varepsilon<\beta_C(C)-2$, there is $\{x_n\}\subseteq C$ such that $\beta_C(C)\le\beta_D(\{x_n\})+\varepsilon$, where $D=\overline{\mathrm{co}}\{x_n\}$ (Lemma 3.2.4). There is a β-minimal subsequence $\{y_n\}$ of $\{x_n\}$ such that

$\beta_D(\{y_n\})=\beta_D(\{x_n\})$ (Proposition 1.3.8) and $\lim_{n\to\infty}\|y_n-z\|$ exists for all $z\in D$ (Lemma 3.3.2). Then

$$\beta_C(C)\le\beta_D(\{y_n\})+\varepsilon=2\inf_{z\in D}\left\{\lim_{n\to\infty}\|y_n-z\|\right\}+\varepsilon\le 2(1+\inf_{z\in D}\|z\|)+\varepsilon.$$

From Theorem 3.2.5, $\delta(D)\ge N_\beta(X)\beta_D(D)/2>N_\beta(X)$, then $S_X(N_\beta(X))\le 1-\inf_{z\in D}\|z\|$. Hence $\beta_C(C)\le 2\big(2-S_X(N_\beta(X))\big)+\varepsilon$, the result follows since ε and C are arbitrary.

Proposition 5.1.17. *If X and Y are isomorphic Banach spaces, then $R_\beta(X)\le d(X,Y)R_\beta(Y)$.*

Proof. Let C be any closed, convex subset of $\overline{B}_X$. For any $\varepsilon>0$, let $U:X\longrightarrow Y$ be an isomorphism such that $\|U\|=1$ and $\|U^{-1}\|\le d(X,Y)+\varepsilon$. Then $U(C)\subseteq\overline{B}_Y$ is closed and convex. Hence

$$\begin{aligned}\beta_C(C)&=\beta_{U^{-1}U(C)}(U^{-1}U(C))\\&\le\|U^{-1}\|\beta_{U(C)}(U(C))\\&\le 2R_\beta(Y)(d(X,Y)+\varepsilon).\end{aligned}$$

Therefore $R_\beta(X)\le(d(X,Y)+\varepsilon)R_\beta(Y)$ since C is arbitrary, and the result follows by letting $\varepsilon\longrightarrow 0$.

Example 5.1.18. Let $l_{p,\infty}$ $(1<p<\infty)$ be the space defined in example 4.3.6, then $d(l_{p,\infty},l^p)\le 2^{1/p}$. Hence $R_\beta(l_{p,\infty})\le 2^{1/p}$, and for $T\in\mathscr{L}(l_{p,\infty},l^r)$ $(1<r<\infty)$, $2^{-1/p}\beta(T)\le\beta(T^*)\le\beta(T)$.

5.2. A seminorm related to $\beta(T^*)$

First we give the definition of a seminorm $\lambda(T)$ which was defined in

[LeS] and [Se], see also [EdE], p.24 .

Definition 5.2.1. Let X and Y be Banach spaces and let $T\in\mathscr{L}(X, Y)$. Define the seminorm $\lambda(T)$ of T as:

$$\lambda(T)=\inf\left\{\|T|_{L}\| : L \text{ is a subspace of } X \text{ with finite codimension}\right\},$$

where $\|T|_{L}\|=\sup\limits_{x\in L\setminus\{0\}}\|Tx\|/\|x\|$.

$\lambda(T)$ is indeed a seminorm on $\mathscr{L}(X, Y)$, see [EdE], p.24, for the verification. Sedeav [Se] proved $\lambda(T)=\beta(T^*)$ for $T\in\mathscr{L}(X)$. Using the same proof, we can obtain the result for $T\in\mathscr{L}(X, Y)$. We give a proof here for completeness.

Theorem 5.2.2. *If X and Y are two Banach spaces and $T\in\mathscr{L}(X, Y)$, then $\lambda(T)=\beta(T^*)$.*

Proof. First we prove that $\lambda(T)\leq\beta(T^*)$. Let $\overline{\mathrm{B}}^*_{\mathrm{Y}}$ be the closed unit ball of Y^*. For any $a>\beta(T^*)=\beta(T^*\overline{\mathrm{B}}^*_{\mathrm{Y}})/2$, there are $x^*_i\in X^*$, $i=1, \dots, n$, so that

$$T^*\overline{\mathrm{B}}^*_{\mathrm{Y}}\subseteq\bigcup_{i=1}^{n}\mathrm{B}(x^*_i, a).$$

Let $L=\left\{x\in X:\ x^*_i(x)=0,\ i=1, \dots, n\right\}$, then L has finite codimension. we will prove $\|T|_{L}\|\leq a$. For any $x\in L$, there is $y^*\in Y^*$, $\|y^*\|=1$ such that $\|Tx\|=y^*(Tx)=(T^*y^*, x)$. For this T^*y^*, there exists x^*_i $(1\leq i\leq n)$ such that $\|T^*y^*-x^*_i\|\leq a$. Hence $\|Tx\|=(T^*y^*-x^*_i, x)+(x^*_i, x)\leq\|x\|\,\|T^*y^*-x^*_i\|\leq a\|x\|$. Therefore $\|T|_{L}\|\leq a$. By the definition of $\lambda(T)$, we obtain $\lambda(T)\leq a$. By the arbitrariness of a, we have $\lambda(T)\leq\beta(T^*)$.

Conversely we prove $\lambda(T)\geq\beta(T^*)$. Let L be any subspace of X with finite codimension and let $\|T|_{L}\|=b$, we will show that $\beta(T^*)\leq b$. For any $y^*\in\overline{\mathrm{B}}^*_{\mathrm{Y}}$, we have

$$\|T^*y^*|_L\| = \sup_{x\in L\setminus\{0\}} |(T^*y^*)(x)|/\|x\| = \sup_{x\in L\setminus\{0\}} |y^*(Tx)|/\|x\|$$

$$\leq \|y^*\| \sup_{x\in L\setminus\{0\}} \|Tx\|/\|x\| \leq \|T|_L\| = b.$$

By the Hahn-Banach theorem, there is $x_1^*\in X^*$ such that $\|x_1^*\|\leq b$ and $x_1^*(x)=(T^*y^*)(x)$ for any $x\in L$. Hence $T^*y^*=x_1^*+x_2^*$ with $x_2^*\in L^{\perp}$, where

$$L^{\perp}=\{x^*\in X^* : x^*(x)=0 \text{ for every } x\in L\}.$$

Therefore $T^*\overline{B}_Y^*\subseteq b\overline{B}_X^*+M$ with $M\subseteq L^{\perp}$ bounded. Since $L^{\perp}$ is a finite-dimensional space, $\beta(M)=0$. Now we obtain $\beta(T^*\overline{B}_Y^*)\leq 2b$, hence $\beta(T^*)\leq b$.

[LeS] gave the inequalities $\lambda(T)/8\leq\lambda(T^*)\leq 8\lambda(T)$ between $\lambda(T)$ and $\lambda(T^*)$. However as a consequence of Theorems 5.1.2, 5.2.2 and Corollary 5.1.7, we have the following result relating $\lambda(T)$ and $\lambda(T^*)$.

Corollary 5.2.3. *Let X and Y be two Banach spaces and let $T\in\mathscr{L}(X, Y)$. Then we have $\lambda(T)/R_\beta(Y^*)\leq\lambda(T^*)\leq R_\beta(X^{**})\lambda(T)$. In particular, if $T\in\mathscr{L}(H_1, H_2)$, where H_1 and H_2 are Hilbert spaces, or $T\in\mathscr{L}(l^p, l^r)$ $(1<p, r<\infty)$, $\lambda(T^*)=\lambda(T)$.*

Now we give a relation between $R_\beta(X)$ and $R_\beta(X^{**})$.

Propoaition 5.2.4. *For Banach space X, $R_\beta(X)\leq R_\beta(X^{**})$. If X is a reflexive Banach space, $R_\beta(X)=R_\beta(X^{**})$.*

Proof. For any closed and convex subset C of $\overline{B}_X$, J_XC is a closed and convex subset of $\overline{B}_X^{**}$, the closed unit ball of X^{**}. In fact, for any $\{J_Xx_n\}\subseteq J_XC$ so that $J_Xx_n\to x^{**}$ $(n\to\infty)$, $\{x_n\}\subseteq C$ is a Cauchy sequence, since

$\|J_X x_n - J_X x_m\| = \|x_n - x_m\|$. Then there is $x \in X$ suct that $x_n \longrightarrow x$ $(n \longrightarrow \infty)$. Since C is closed, $x \in C$. Therefore $x^{**} = J_X x \in J_X C$. Also $a J_X x + b J_X y = J_X(ax+by)$ implies the convexity of $J_X C$. Obviously $\beta_C(C) = \beta_{J_X C}(J_X C)$. So $R_\beta(X) \leq R_\beta(X^{**})$.

Remark. The inequality can be strict. For example this is so in (c_0), since $R_\beta(c_0) \leq 4/3$ and $R_\beta(l^\infty) = 2$ $((c_0)^{**} = l^\infty)$ as examples in section 5.1 show.

5.3. The ascents, descents and eigenvalues of operators in $\mathscr{L}(X)$

Let X be a Banach space and let $T \in \mathscr{L}(X)$. We use $\mathscr{N}(T) := \{x \in X: Tx = 0\}$ to denote the *null space* of T, and $\mathscr{R}(T) := \{y \in X$: there is $x \in X$ so that $Tx = y\}$ the *range space* of T. Obviously we have

$$\{0\} = \mathscr{N}(T^0) \subseteq \mathscr{N}(T) \subseteq \mathscr{N}(T^2) \subseteq \mathscr{N}(T^3) \subseteq \ldots\ldots$$

and

$$X = \mathscr{R}(T^0) \supseteq \mathscr{R}(T) \supseteq \mathscr{R}(T^2) \supseteq \mathscr{R}(T^3) \supseteq \ldots\ldots,$$

where $T^0 = I$, the identity mapping. If $\mathscr{N}(T^n) = \mathscr{N}(T^{n+1})$ for some $n \in \mathbb{N} \cup \{0\}$, then $\mathscr{N}(T^n) = \mathscr{N}(T^{n+i})$ for all $i \in \mathbb{N}$. Also $\mathscr{R}(T^m) = \mathscr{R}(T^{m+1})$ for some $m \in \mathbb{N} \cup \{0\}$ implies that $\mathscr{R}(T^m) = \mathscr{R}(T^{m+i})$ for all $i \in \mathbb{N}$. Recall that (cf. [TL], p.290) the smallest number $n \in \mathbb{N} \cup \{0\}$ such that $\mathscr{N}(T^n) = \mathscr{N}(T^{n+1})$ is called the *ascent* of T, denoted by $asc(T)$; and the smallest number $m \in \mathbb{N} \cup \{0\}$ such that $\mathscr{R}(T^m) = \mathscr{R}(T^{m+1})$ is called the *descent* of T, denoted by $desc(T)$. If no such n or m exists, we set $asc(T) = +\infty$ or $desc(T) = +\infty$. We know that if $asc(T) = n < +\infty$ and $desc(T) = m < +\infty$, then $asc(T) = desc(T) = n$ and $X = \mathscr{N}(T^n) \oplus \mathscr{R}(T^n)$ (see [TL], p.290).

Theorem 5.3.1. *For $T \in \mathscr{L}(X)$, if $\alpha(T) < 1$ or $\beta(T) < 1$, then*

$$asc(I-T) = desc(I-T) = n < +\infty \quad \textit{and} \quad X = \mathscr{N}(I-T)^n \oplus \mathscr{R}(I-T)^n.$$

Proof. Suppose $\alpha(T)<1$. We only need to prove $asc(I-T)<+\infty$ and $desc(I-T)<+\infty$. Put $\mathscr{N}_n=\mathscr{N}(I-T)^n$, $n\in\mathbb{N}\cup\{0\}$, then each $\mathscr{N}_n$ is a closed subspace of X, $\mathscr{N}_n\subseteq\mathscr{N}_{n+1}$ and $\mathscr{N}_{n+1}=\{x\in X:\ (I-T)x\in\mathscr{N}_n\}$. If $\mathscr{N}_n\neq\mathscr{N}_{n+1}$ for all n, then $\mathscr{N}_n$ is a proper, closed subspace of $\mathscr{N}_{n+1}$. Let $0<\gamma<1$. By Riesz's Lemma (cf. [Mar], p.84, Lemma 5.1), there is $x_{n+1}\in\mathscr{N}_{n+1}$, $\|x_{n+1}\|=1$ so that $\mathrm{dist}(x_{n+1}, \mathscr{N}_n)\geq\gamma$. It is easy to see that if $x\in\mathscr{N}_{n+1}$ and $\mathrm{dist}(x, \mathscr{N}_n)\geq\gamma$, then $Tx\in\mathscr{N}_{n+1}$ and $\mathrm{dist}(Tx, \mathscr{N}_n)\geq\gamma$. In fact, $(I-T)^{n+1}(Tx)=T\big((I-T)^{n+1}x\big)=0$; and for any $y\in\mathscr{N}_n$, noting that $(I-T)x\in\mathscr{N}_n$, we have

$$\|Tx-y\|=\|x-\big((I-T)x+y\big)\|\geq\mathrm{dist}(x, \mathscr{N}_n)\geq\gamma.$$

Now we see that for each n and all $p\in\mathbb{N}$, $T^px_{n+1}\in\mathscr{N}_{n+1}$ and $\mathrm{dist}(T^px_{n+1}, \mathscr{N}_n)\geq\gamma$. If $k<n$, $T^px_{k+1}\in\mathscr{N}_{k+1}\subseteq\mathscr{N}_n$, so $\|T^px_{n+1}-T^px_{k+1}\|\geq\gamma$. Thus we obtain $\alpha(\{T^px_{n+1}\}_{n=0}^{\infty})\geq\gamma$ for every p. However

$$\alpha(\{T^px_{n+1}\}_{n=0}^{\infty})\leq\{\alpha(T)\}^p\alpha(\{x_{n+1}\}_{n=0}^{\infty})\leq 2\{\alpha(T)\}^p\longrightarrow 0\ \ (p\longrightarrow\infty),$$

as $\alpha(T)<1$. This is a contradiction. Hence $asc(I-T)<+\infty$.

Similarly, we show $desc(I-T)<+\infty$. Let $\mathscr{R}_m=\mathscr{R}(I-T)^m$, $m\in\mathbb{N}\cup\{0\}$, Then $\mathscr{R}_{m+1}\subseteq\mathscr{R}_m$, $\mathscr{R}_{m+1}=(I-T)\mathscr{R}_m$. Also for any m, $\mathscr{R}_m$ is a closed, linear subspace of X. We only need to prove $\mathscr{R}_1$ is closed. Let $y_k\in\mathscr{R}_1$ and $y_k\longrightarrow y$ $(k\longrightarrow\infty)$. Take $x_k\in X$ so that $y_k=(I-T)x_k$. Since $x_k=y_k+Tx_k$, we have

$$\alpha(\{x_k\})\leq\alpha(\{y_k\})+\alpha(\{Tx_k\})\leq\alpha(T)\alpha(\{x_k\}).$$

We obtain $\alpha(\{x_k\})=0$ as $\alpha(T)<1$. Hence there is a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $x_{k_j}\longrightarrow x$. Therefore $y=(I-T)x\in\mathscr{R}_1$.

If $\mathscr{R}_m\neq\mathscr{R}_{m+1}$ for every m, then $\mathscr{R}_{m+1}$ is a proper, closed subspace of $\mathscr{R}_m$. Let $0<\gamma<1$. For each m, there is $x_m\in\mathscr{R}_m$, $\|x_m\|=1$ so that $\mathrm{dist}(x_m, \mathscr{R}_{m+1})\geq\gamma$. We can show that if $x\in\mathscr{R}_m$ and $\mathrm{dist}(x, \mathscr{R}_{m+1})\geq\gamma$, then $Tx\in\mathscr{R}_m$ and $\mathrm{dist}(Tx, \mathscr{R}_{m+1})\geq\gamma$. In fact, $Tx=x-(I-T)x$, $x\in\mathscr{R}_m$, $(I-T)x\in\mathscr{R}_{m+1}\subseteq\mathscr{R}_m$, so $Tx\in\mathscr{R}_m$. And for any $z\in\mathscr{R}_{m+1}$, noting that $(I-T)x+z\in\mathscr{R}_{m+1}$, we have

$$\|Tx-z\|=\|x-((I-T)x+z)\|\geq \mathrm{dist}(x,\mathscr{R}_{m+1})\geq\gamma.$$

Now we see that for every m and $p\in\mathbb{N}$, $T^p x_m\in\mathscr{R}_m$ and $\mathrm{dist}(T^p x_m, \mathscr{R}_{m+1})\geq\gamma$. If $k>m$, $T^p x_k\in\mathscr{R}_k\subseteq\mathscr{R}_{m+1}$, so $\|T^p x_m - T^p x_k\|\geq\gamma$. Thus we have $\alpha(\{T^p x_m\}_{m=0}^{\infty})\geq\gamma$ for every p. But

$$\alpha(\{T^p x_m\}_{m=0}^{\infty})\leq\{\alpha(T)\}^p\alpha(\{x_m\}_{m=0}^{\infty})\leq 2\{\alpha(T)\}^p\longrightarrow 0\ (p\longrightarrow\infty),$$

as $\alpha(T)<1$. This is a contradiction. Hence $desc(I-T)<+\infty$.

If $\beta(T)<1$, we can also prove every $\mathscr{R}_m$ is closed, and the conclusion can be obtained by noting that $\beta(T^P)<\gamma/4$ (such p exists since $\beta(T)<1$) implies $\alpha(T^P)<\gamma/2$.

Corollary 5.3.2. *Let $T\in\mathscr{L}(X)$ with $\alpha(T)<1$ or $\beta(T)<1$. If $I-T$ is injective, then $I-T$ is surjective and $(I-T)^{-1}\in\mathscr{L}(X)$.*

Proof. Since $I-T$ is injective, $asc(I-T)=0$. By Theorem 5.3.1, we have $desc(I-T)=asc(I-T)=0$. So $\mathscr{R}(I-T)=X$.

Now we show $(I-T)^{-1}$ is continuous. We only give the proof for $\alpha(T)<1$, the proof for $\beta(T)<1$ is similar. It is enough to prove that there is $b>0$ such that $\|(I-T)x\|\geq b\|x\|$ for all $x\in X$. If this is not true, there exists a sequence $\{x_n\}\subseteq X$ so that $\|x_n\|=1$ and $\lim_{n\to\infty}\|(I-T)x_n\|=0$. Hence

$$\alpha(\{x_n\})=\alpha(\{Tx_n+(I-T)x_n\})\leq\alpha(\{Tx_n\})\leq\alpha(T)\alpha(\{x_n\}).$$

We have $\alpha(\{x_n\})=0$ since $\alpha(T)<1$. So there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and an $x\in X$ such that $\lim_{k\to\infty}x_{n_k}=x$. However this implies $\|x\|=1$ and $(I-T)x=0$, which contradicts the assumption that $I-T$ is injective.

Remark. The above result generalizes Lemma 5.2 on page 84 in [Mar], where it is supposed that $\alpha(T)<1/2$.

For $T\in\mathscr{L}(X)$, recall that the *resolvent set* $\rho(T)$ of T is the set

$$\rho(T):=\{\lambda\in\mathbb{R}:\ (\lambda I-T)^{-1}\text{ exists and }(\lambda I-T)^{-1}\in\mathscr{L}(X)\}.$$

The *spectrum* $\sigma(T)$ of T is the set $\mathbb{R}\setminus\rho(T)$. Also $\rho(T)$ is an open set, $\sigma(T)$ is a closed, bounded set. $\lambda\in\mathbb{R}$ is said to be an *eigenvalue* of T if there is an $x_\lambda\in X$, $x_\lambda\neq 0$ such that $Tx_\lambda=\lambda x_\lambda$. The nonzero member x_λ is said to be an *eigenvector* corresponding to the eigenvalue λ. Obviously, λ is an eigenvalue of T if and only if $\lambda I-T$ is not injective. So any eigenvalue of T is in $\sigma(T)$.

Theorem 5.3.3. *Let $T\in\mathscr{L}(X)$. We have*

1) *If $|\lambda|>\alpha(T)$, then $\lambda\in\rho(T)$ or λ is an eigenvalue of T of finite algebraic multiplicity.*

2) *For each $r\geq 0$, let $\sigma_r(T)=\{\lambda\in\sigma(T):\ |\lambda|\geq r\}$. If $r>\alpha(T)$, then $\sigma_r(T)=\emptyset$ or $\sigma_r(T)$ is a finite set of eigenvalues of T.*

Proof. If $|\lambda|>\alpha(T)$, then $\alpha(\lambda^{-1}T)=|\lambda|^{-1}\alpha(T)<1$. Since $\lambda I-T=\lambda(I-\lambda^{-1}T)$, $\lambda I-T$ is injective if, and only if, $I-\lambda^{-1}T$ is injective. So by Corollary 5.3.2, $\lambda\in\rho(T)$ (if $I-\lambda^{-1}T$ is injective), or λ is an eigenvalue of T (if $I-\lambda^{-1}T$ is not injective). By Theorem 5.3.1, there is $n\in\mathbb{N}\cup\{0\}$ so that

$$\mathscr{N}(I-\lambda^{-1}T)^n=\mathscr{N}(I-\lambda^{-1}T)^{n+1},$$

then $\mathscr{N}(\lambda I-T)^n=\mathscr{N}(\lambda I-T)^{n+1}$. We have proved the conclusion 1).

To show 2), we only need to prove that for $r>\alpha(T)$, if $\sigma_r(T)\neq\emptyset$, then $\sigma_r(T)$ is finite, since $\sigma_r(T)$ contains only eigenvalues of T. If $\sigma_r(T)$ is not a finite set, then there is a sequence $\{\lambda_k\}$ of distinct eigenvalues of T with $|\lambda_k|\geq r$. For each k, let x_k be a nonzero eigenvector corresponding to λ_k. Then the set $\{x_1, \dots, x_k\}$ is linearly independent for each $k \ge 1$. This is obviously true for $k=1$. Assuming that this is true for some $k \ge 1$ and not true for $k+1$, implies that

$$x_{k+1} = \sum_{i=1}^{k} \alpha_i x_i \quad \text{with} \quad \sum_{i=1}^{k} |\alpha_i| > 0.$$

Then we have

$$0 = Tx_{k+1} - \lambda_{k+1} x_{k+1} = \sum_{i=1}^{k} \alpha_i Tx_i - \lambda_{k+1} \sum_{i=1}^{k} \alpha_i x_i = \sum_{i=1}^{k} \alpha_i (\lambda_i - \lambda_{k+1}) x_i .$$

Thus $\alpha_i = 0$ for all $i = 1, \dots, k$, since $\{x_1, \dots, x_k\}$ is linearly independent and $\lambda_i \ne \lambda_{k+1}$ for $i \ne k+1$. This contradicts $\sum_{i=1}^{k} |\alpha_i| > 0$. Hence $\{x_1, \dots, x_{k+1}\}$ must be linearly independent. We have proved $\{x_1, \dots, x_k\}$ is linearly independent for each $k \ge 1$ by induction. For each $k \ge 1$, let $M_k = \text{span}\{x_1, \dots, x_k\}$, and let $0 < \gamma < 1$. For each $k \ge 2$, since M_{k-1} is a closed, proper subspace of M_k, we see that there is $z_k \in M_k$, $\|z_k\| = 1$ such that $\text{dist}(z_k, M_{k-1}) \ge \gamma$. For every $x = \sum_{i=1}^{k} \alpha_i x_i \in M_k$, we have

$$(\lambda_k I - T)x = \sum_{i=1}^{k} (\lambda_k \alpha_i x_i - \alpha_i \lambda_i x_i) = \sum_{i=1}^{k-1} \alpha_i (\lambda_k - \lambda_i) x_i \in M_{k-1} ,$$

and $Tx = \sum_{i=1}^{k} \alpha_i \lambda_i x_i \in M_k$. Hence if $x \in M_k$ and $\text{dist}(x, M_{k-1}) \ge \gamma$, then $\lambda_k^{-1} Tx \in M_k$, and $\text{dist}(\lambda_k^{-1} Tx, M_{k-1}) \ge \gamma$, as for any $y \in M_{k-1}$, we have

$$\|\lambda_k^{-1} Tx - y\| = \|x - (\lambda_k^{-1}(\lambda_k I - T)x + y)\| \ge \text{dist}(x, M_{k-1}) \ge \gamma.$$

Now we see that $\lambda_k^{-p} T^p z_k \in M_k$ and $\text{dist}(\lambda_k^{-p} T^p z_k, M_{k-1}) \ge \gamma$ for each $k \ge 2$ and all $p \in \mathbb{R}$. In particular, if $2 < j < k$, since $T^p z_j \in M_j \subseteq M_{k-1}$, we have

$$\|(r^{-1}T)^p z_k - (r^{-1}T)^p z_j\| \ge \|\lambda_k^{-p} T^p z_k - \lambda_k^{-p} T^p z_j\| \ge \gamma.$$

Hence $\alpha(\{(r^{-1}T)^p z_k\}_{k=2}^{\infty}) \ge \gamma$ for all p. But

$$\alpha(\{(r^{-1}T)^p z_k\}_{k=2}^{\infty}) \le (r^{-1}\alpha(T))^p \alpha(\{z_k\}_{k=2}^{\infty}) \le 2(r^{-1}\alpha(T))^p \longrightarrow 0 \ (p \to \infty),$$

since $r > \alpha(T)$. This contradiction shows that $\sigma_r(T)$ must be finite and the proof is complete.

Remark. The theorem is also true if α is replaced with β. This result extends Proposition 5.8 on page 85 in [Mar] where $r > 2\alpha(T)$ was needed, and gives a positive answer to the question asked by Martin after that proposition.

Let $r(T) = \lim_{n\to\infty} [\alpha(T^n)]^{1/n} = \lim_{n\to\infty} [\beta(T^n)]^{1/n}$. Obviously $r(T) \leq \alpha(T)$ and $r(T) \leq \beta(T)$. We can use $r(T) < 1$ to replace $\alpha(T) < 1$ (or $\beta(T) < 1$) in Theorem 5.3.1 and Corollary 5.3.2, use $|\lambda| > r(T)$ to replace $|\lambda| > \alpha(T)$ in Theorem 5.3.3. 1), and use $r > r(T)$ to replace $r > \alpha(T)$ in Theorem 5.3.3. 2). The proofs are similar. Nussbaum [Nu-1] proved that $r(T) = r_e(T)$, where $r_e(T)$ denotes the radius of the essential spectrum (for precise definition, see [Nu-1]). Note that our results imply that $r_e(T) \leq r(T)$.

REFERENCES

[Am] D. Amir, On Jung's constant and related constants in normed linear spaces, *Pacific J. Math.* 118(1985), 1-15.

[Ari] J. Arias de Reyna, On r-separated sets in normed spaces, *Proc. Amer. Math. Soc.* 112(1991), 1087-1094.

[AroP] N. Aronszajn and P. Panitchpakdi, Extension of uniformly continuous transformations and hyperconvex metric spaces, *Pacific J. Math.* 6(1956), 405-439.

[Ban] J. Banas, Compactness conditions in the geometric theory of Banach spaces, *Nonlinear Anal. TMA*, 16(1991), 669-682.

[Bar] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Nordhoff International Publishing, Leyden, Netherlands, 1976.

[Bea] B. Beauzamy, *Introduction to Banach spaces and their geometry*, Amsterdam, North-Holland, 1982.

[Ben-1] T. D. Benavides, Some properties of the set and ball measures of noncompactness and applications, *J. London Math. Soc.*, (2) 34(1986), 120-128.

[Ben-2] T. D. Benavides, Set-contractions and ball-contractions in some classes of spaces, *J. Math. Anal. Appl.*, 136(1988), 131-140.

[Ben-3] T. D. Benavides, Normal structure coefficients of $L^P(\Omega)$, *Proc. Roy. Soc. Edinburgh, Section A*, 117(1991), 299-303.

[BenA] T. D. Benavides and J. M. Ayerbe, Set contractions and ball contractions in L^p-spaces, *J. Math. Anal. Appl.* (to appear).

[BenL-1] T. D. Benavides and G. Lopez Acedo, Connections between some measures of noncompactness and associated operators, *Extracta Math.*, 5(1990), 62-64.

[BenL-2] T. D. Benavides and G. Lopez Acedo, Moduli of noncompact convexity and lower bounds for normal structure coefficients, *Proc. Roy. Soc. Edinburgh, Section A* (to appear).

[Bo] F. Bohnenblust, Convex regions and projections in Minkowski spaces, *Ann. Math.*, 39(1938), 301-308.

[Bro] F. E. Browder, Nonexpansive nonlinear operators in a Banach space, *Proc. Nat. Acad. Sci. U.S.A.*, 54(1965), 1041-1044.

[Bru] R. E. Bruck, Asymptotic behavior of nonexpansive mappings, *Contemporary Math.* vol. 18 (*Fixed Points and Nonexpansive Mappings*), p. 1–47, *Amer. Math. Soc.*, Providence, RIm, 1983.

[By-1] W. L. Bynum, A class of spaces lacking normal structure, *Compositio Math.*, 25(1972), 233-236.

[By-2] W. L. Bynum, Normal structure coefficients for Banach spaces, *Pacific J. Math.*, 86(1980), 427-436.

[CaM] E. Casini and E. Maluta, Fixed points of uniformly Lipschitzian mappings in spaces with uniformly normal structure, *Nonlinear Anal. TMA* 9(1985), 103-108.

[Cl] J. A. Clarkson, Uniformly convex spaces, *Trans. Amer. Math. Soc.* 40(1936), 396–414.

[Dan-1] J. Danes, On densifying and related mappings and their application in nonlinear functional analysis, *Theory of Nonlinear Operators*, Proceedings of a summer-school, Akademie-Verlag, Berlin, 1974, p. 15-56.

[Dan-2] J. Danes, On the radius of a set in a Hilbert space, *Comment. Math. Univ. Carolin.*, 25(1984), 335-362.

[Day] M. M. Day, Uniform convexity in factor and conjugate spaces, *Ann. Math.*, 45(1944), 375–385.

[De] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[DoI] W. G. Dotson, JR., On the Mann iterative process, *Trans. Amer. Math. Soc.*, 149(1970), 65-73

[DowT] D. J. Downing and B. Turett, Some properties of the characteristic of convexity relating to fixed point theory, *Pacific J. Math.*, 104(1983), 343-350.

[EdE] D. E. Edmunds and W. D. Evans, *Spectral Theory and Differential Operators*, Clarendon Press, Oxford, 1987.

[ElO] J. Elton and E. Odell, The unit ball of every infinite dimensional normed linear space contains a $(1+\varepsilon)$-separated sequence, *Colloquium Math.*, 44(1981), 105-109.

[GaL] J. Gao and K. Lau, On two classes of Banach spaces with uniform normal structure, Studia Math., 99(1991), 41–56.

[GoeK-1] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, London, 1990.

[GoeK-2] K. Goebel and W. A. Kirk, A fixed point theorem for transformations whose iterates have uniform Lipschitz constant, *Studia Math.* 47(1973), 135–140.

[GoeKT] K. Goebel, W. A. Kirk and R. L. Thele, Uniformly Lipschitzian families of transformations in Banach spaces, *Can. J. Math.* 26(1974), 1245–1256.

[GoeS] K. Goebel and T. Sekowski, The modulus of noncompact convexity, *Ann. Univ. Mariae Curie-Sklodowska*, 38A(1984), 41–48.

[Goh] D. Gohde, Zum prinzip der kontrahierenden abbildung, *Math. Nachr.*, 30(1965), 251–258.

[GolM] L. S. Goldenstein and A. S. Markus, On a measure of noncompactness of bounded sets and linear operators, *Studies in Algebra and Math. Anal.* (Russian), 45–54. Izdat, Karta Moldovenjaski, Kishinev, 1965. (MR, 35(1968), 789).

[GorK] J. Gornicki and M. Kruppel, Fixed points of uniformly Lipschitzian mappings, *Bull. Polish Acad. Sci. Math.*, 36(1988), 57-62.

[GosD] J. P. Gossez and E. Lami Dozo, Some geometric properties related to the fixed point theory for nonexpansive mappings, *Pacific J. Math.*, 40(1972), 565-573.

[Ha] O. Hanner, On the uniformly convexity of L^p and l^p, *Ark. Math.* 3(1956), 239-244.

[Ho] R. B. Holmes, A Course on Optimization and Best Approximation, *Lecture Notes in Mathematics*, vol.275, Springer-Verlag, Berlin, 1972.

[Ja] R. C. James, A non-reflexive Banach space isometric with its second conjugate space, *Proc. Nat. Acad. Sci. U.S.A.*, 37(1951), 174-177.

[Ju] H. W. E. Jung, Über die kleinste Kugel, die eine räumliche Figur einschlisst, *J. Reine Angew. Math.*, 123(1901), 241-257.

[Ke] J. L. Kelley, Banach space with the extension property, *Trans. Amer. Math. Soc.*, 72(1952), 323-326.

[Ki] W. A. Kirk, A fixed point theorem for mappings which do not increase distances, *Amer. Math. monthly*, 72(1965), 1004-1006.

[Ko] C. A. Kottman, Packing and reflexivity in Banach spaces, *Trans. Amer. Math. Soc.*, 150(1970), 565-576.

[KrZ] M. A. Krasnosel'skii and P. P. Zabreiko, *Geometrical Methods of Nonlinear Analysis*, Springer-Verlag, Berlin, 1984.

[La] H. E. Lacey, The Isometric Theory of Classical Banach Spaces, Springer-Verlag, Berlin, 1974.

[LeS] A. Lebow and M. Schechter, Semigroups of operators and measures of noncompactness, *J. Functional Anal.*, 7(1971), 1-26.

[Lif] E. A. Lifschitz, Fixed point theorems for operators in strongly convex spaces, *Voronez Gos. Univ. Trudy Mat. Fak.*, 16(1975), 23-28. (Russian)

[Lim-1] T. C. Lim, Fixed point theorems for uniformly Lipschitzian mappings in L^p space, *Nonlinear Anal. TMA*, 7(1983), 555-563.

[Lim-2] T. C. Lim, On the normal structure coefficient and the bounded sequence coefficient, *Proc. Amer. Math. Soc.*, 88(1983), 262-264.

[LinT-1] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces* II, *Function Spaces*, Springer-Verlag, Berlin, 1979.

[LinT-2] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces, *Lecture Notes in Mathematics*, vol.338, Springer-Verlag, 1973.

[Ll] N. G. Lloyd, *Degree Theory*, *Cambridge Tracts in Mathematics* **73**, Cambridge University Press, London, 1978.

[Mal] E. Maluta, Uniformly normal structure and related coefficients, *Pacific J. Math.*, **111**(1984), 357-369.

[Man] W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, 4(1953), 506-510.

[Mar] R. H. Martin, Jr., *Nonlinear Operators and Differential Equations in Banach Spaces*, John Wiley & Sons, New York, 1976.

[Mas] P. Massatt, Some properties of condensing maps, *Ann. Mat. Pura. Appl.*, (4) **125**(1980), 101-115.

[MiM] D. P. Mil'man and V. D. Mil'man, The geometry of nested families with empty intersection —— Structure of the unit sphere of a nonreflexive space, *Amer. Math. Soc. Transl.*, (2) 85(1969), 233-243.

[Mo] V. Montesinos, Drop property equals reflexivity, *Studia Math.*, 87(1987), 93-100.

[Na] L. Nachbin, A theorem of Hahn-Banach type, *Trans. Amer. Math. Soc.*, 68(1950), 28-46.

[Nu-1] R. D. Nussbaum, The radius of the essential spectrum, *Duke Math. J.*, 37(1970), 473-478.

[Nu-2] R. D. Nussbaum, The ball intersection property for Banach spaces, *Bull. Acad. Polon. Sci.*, 19(1971), 931–936.

[O] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.*, 73(1967), 591–597.

[Pa] P. L. Papini, Some parameters of Banach spaces, *Rend. Sem. Mat. Fis. Milano*, 53(1983), 131–148(1986).

[Pi] S. A. Pichugov, Jung's constant of the space L_p, *Mat. Zametki*, 43(1988), 604–614. (Russian)

[Pr] S. Prus, On Bynum's fixed point theorem, *Atti. Sem. Mat. Fis. Univ. Modena*, 38(1990), 535–545.

[Re] S. Reich, Product formulas, nonlinear semigroups, and accretive operators, *J. Functional Anal.*, 36(1980), 147–168.

[Ro-1] S. Rolewicz, On drop property, *Studia Math.*, 85(1987), 27–35.

[Ro-2] S. Rolewicz, On Δ-uniform convexity and drop property, *Studia Math.*, 87(1987), 181–191.

[Se] A. A. Sedaev, The structure of certain linear operators, *Mat. Issled*, 5(1970), *vyp.* 1(15), 166–175 (Russian).

[Sm] R. Smarzewski, On an inequality of Bynum and Drew, *J. Math. Anal. Appl.*, 150(1990), 146–150.

[TL] A. E. Taylor and D. C. Lay, *Introduction to Functional Analysis*, 2nd ed., John Wiley & Sons, New York, 1980.

[Web-1] J. R. L. Webb, Remarks on k-set contractions, *Boll. Un. Mat. Ital.*, (4) 4(1971), 614–629.

[Web-2] J. R. L. Webb, On seminorms of operators, *J. London Math. Soc.*, (2) 7(1973), 337–342.

[Web-3] J. R. L. Webb, Approximation solvability of nonlinear equations, *Nonlinear Analysis, Function Spaces and Applications*, vol.2, Teubner–Texte zur

Mathematik, Leipzig, 1982, p. 234–257.

[Web-4] J. R. L. Webb, On a property of duality mappings and the A-properness of accretive operators, *Bull. London Math. Soc.*, 13(1981), 235–238.

[Web-5] J. R. L. Webb, Mappings of accretive and pseudo-A-proper type, *J. Math. Anal. Appl.*, 85(1982), 146–152.

[WebZ-1] J. R. L. Webb and W. Zhao, An L^P inequality, *Proc. Roy. Soc. Edinburgh*, 116A(1990), 129–131.

[WebZ-2] J. R. L. Webb and W. Zhao, On connections between set and ball measures of noncompactness, *Bull. London Math. Soc.*, 22(1990), 471–477.

[WelW] J. H. Wells and L. R. Williams, *Embeddings and Extensions in Analysis*, Springer-Verlag, Berlin, 1975.

[XX] H. Xu and Z. Xu, An L^P inequality and its applications to fixed point theory and approximation theory, *Proc. Roy. Soc. Edinburgh*, 112A(1989), 343–351.

[Z-1] W. Zhao, Remarks on various measures of noncompactness, *J. Math. Anal. Appl.* (to appear).

[Z-2] W. Zhao, Normal structure coefficients and seminorms of linear operators in Banach spaces, *Nonlinear Anal. TMA* (to appear).